Time-dependent angularly averaged inverse transport (extended version)

Guillaume Bal and Alexandre Jollivet *

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Abstract

This paper concerns the reconstruction of the absorption and scattering parameters in a time-dependent linear transport equation from knowledge of angularly averaged measurements performed at the boundary of a domain of interest. We show that the absorption coefficient and the spatial component of the scattering coefficient are uniquely determined by such measurements. We obtain stability results on the reconstruction of the absorption and scattering parameters with respect to the measured albedo operator. The stability results are obtained by a precise decomposition of the measurements into components with different singular behavior in the time domain.

1 Introduction

Inverse transport theory has many applications in e.g. medical and geophysical imaging. It consists of reconstructing optical parameters in a domain of interest from measurements of the transport solution at the boundary of that domain. The optical parameters are the total absorption (extinction) parameter $\sigma(x)$ and the scattering parameter $k(x, v', v)$, which measures the probability of a particle at position $x \in X \subset \mathbb{R}^n$ to scatter from direction $v' \in \mathbb{S}^{n-1}$ to direction $v \in \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$.

The domain of interest is probed as follows. A known flux of particles enters the domain and the flux of outgoing particles is measured at the domain's boundary. Several inverse theories may then be envisioned based on available data. In this paper, we assume availability of time dependent measurements that are angularly averaged. Also the source term used to probe the domain is not resolved angularly in order to e.g. save time in the acquisition of data. More precisely, the incoming density of particles $\phi(t, x, v)$ as a function of time t, at position $x \in \partial X$ at the boundary of the domain of interest, and for incoming directions v, is of the form $\phi_S(t, x, v) = \phi(t, x)S(x, v)$, where $\phi(t, x)$ is arbitrary but $S(x, v)$ is fixed. This paper is concerned with the reconstruction of the optical parameters from such measurements. We show that the attenuation coefficient is uniquely determined and that the spatial structure of the scattering coefficient can be reconstructed provided that scattering vanishes in the vicinity of the domain's boundary (except in dimension $n = 2$ and when X is a disc, where our theory does not require k to vanish in the vicinity of ∂X). For instance, when $k(x, v', v) = k_0(x)g(v', v)$ with $g(v', v)$ known a priori, then $k_0(x)$ is uniquely determined by the measurements. Similar

*Department of Applied Physics and Applied Mathematics, Columbia University, New York NY, 10027; gb2030@columbia.edu and aj2315@columbia.edu

results were announced in [1] when measurements are available in the modulation frequency variable, which is the dual (Fourier) variable to the time variable.

Several other regimes have been considered in the literature. The uniqueness of the reconstruction of the optical parameters from knowledge of angularly resolved measurements both in the time-dependent and time-independent settings was proved in [9, 10]; see also [17] for a review. Stability in the time-independent case has been analyzed in dimension $n = 2, 3$ under smallness assumptions for the optical parameters in [14, 15] and in dimension $n = 2$ in [18]. Stability results in the presence of full, angularly resolved, measurements have been obtained in [3, 4, 19]. The intermediate case of angularly averaged measurements with angularly resolved sources was considered in [13]. The lack of stability of the reconstruction in the time independent setting with angularly averaged measurements and isotropic sources is treated in [6]. See also [2] for a recent review of results in inverse transport theory.

The rest of the paper is structured as follows. Section 2 recalls known results on the transport equation and the decomposition of the albedo operator. In section 3 we define and decompose the averaged albedo operator (Proposition 3.1) and we study its distributional kernel (Theorems 3.2–3.5). Our main results on uniqueness and stability are presented in section 4 (Theorems 4.1–4.2, Theorems 4.4–4.5 and Corollary 4.6). We show that the absorption coefficient and the spatial structure of the scattering coefficient (the phase function describing scattering from v to v' has to be known in advance) can be reconstructed stably from angularly averaged time dependent data. The reconstruction of the scattering coefficient requires invertion of a weighted Radon transform in the general case. In the specific case of a spherical geometry (measurements are performed at the boundary of a sphere), then the scattering coefficient may be obtained by inverting a *classical* Radon transform. In section 5 we prove Theorems 3.4–3.5. In section 6 we prove Theorems 4.1–4.2, Theorem 4.5 and Theorem 4.4 (4.13). In section 7 we prove Theorem 3.2. In section 8 we prove Theorem 3.3. In section 9 we prove Lemmas 8.1–8.4 that are used in section 8. In section 10 we prove Proposition 3.1.

The derivation of the results is fairly technical and is based on a careful analysis of the temporal behavior of the decomposition of the albedo operator into components that are multilinear in the scattering coefficient. Our results are based on showing that the ballistic and single scattering components may be separated from the rest of the data. These two components are then used to obtain our uniqueness and stability results. It turns out that the structure of single scattering is different depending on whether k vanishes on ∂X or not. When k does not vanish on ∂X, the main singularities of the single scattering component do not allow us to "see inside" the domain as they only depend on values of k at the domain's boundary in dimension $n \geq 3$. The singular structure of single scattering and the resulting stability estimates are presented in detail when both k vanishes and does not vanish on ∂X.

This is the extended version of a submitted paper [5].

2 The forward problem and albedo operator

2.1 The linear Boltzmann transport equation

We now introduce notation and recall some known results on the linear transport equation. Let X be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, with a C^1 boundary ∂X. Let $\nu(x)$ denote the outward normal unit vector to ∂X at $x \in \partial X$. Let $\Gamma_\pm = \{(x,v) \in \partial X \times \mathbb{S}^{n-1} \mid \pm\nu(x)\cdot v > 0\}$ be the sets of incoming and outgoing conditions. For $(x,v) \in \bar{X} \times \mathbb{S}^{n-1}$ we define $\tau_\pm(x,v)$ and $\tau(x,v)$ by $\tau_\pm(x,v) := \inf\{s \in (0,+\infty) \mid x \pm sv \notin X\}$ and $\tau(x,v) := \tau_-(x,v) + \tau_+(x,v)$. For $x \in \partial X$ we define $\mathbb{S}^{n-1}_{x,\pm} := \{v \in \mathbb{S}^{n-1} \mid \pm\nu(x)\cdot v > 0\}$.

Consider $\sigma : X \times \mathbb{S}^{n-1} \to \mathbb{R}$ and $k : X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{R}$ two nonnegative measurable functions. We assume that (σ, k) is admissible when

$$\begin{aligned}
&0 \le \sigma \in L^\infty(X \times \mathbb{S}^{n-1}),\\
&0 \le k(x, v', .) \in L^1(\mathbb{S}^{n-1}) \text{ for a.e. } (x, v') \in X \times \mathbb{S}^{n-1},\\
&\sigma_p(x, v') = \int_{\mathbb{S}^{n-1}} k(x, v', v)dv \text{ belongs to } L^\infty(X \times \mathbb{S}^{n-1}).
\end{aligned} \tag{2.1}$$

Let $T > \eta > 0$. We consider the following linear Boltzmann transport equation

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t, x, v) + v \cdot \nabla_x u(t, x, v) + \sigma(x, v)u(t, x, v)\\
&= \int_{\mathbb{S}^{n-1}} k(x, v', v)u(t, x, v')dv',\ (t, x, v) \in (0, T) \times X \times \mathbb{S}^{n-1},\\
&u_{|(0,T)\times\Gamma_-}(t, x, v) = \phi(t, x, v),\\
&u(0, x, v) = 0,\ (x, v) \in X \times \mathbb{S}^{n-1},
\end{aligned} \tag{2.2}$$

where $\phi \in L^1((0, T), L^1(\Gamma_-, d\xi))$ and $\mathrm{supp}\phi \subseteq [0, \eta]$. Here, $d\xi(x, v) = |v \cdot \nu(x)|dvd\mu(x)$, where $d\mu$ is the surface measure on ∂X and dv is the surface measure on $\mathbb{S}^{n-1}$. In other words, we assume that the initial condition is concentrated in the η-vicinity of $t = 0$ and measurements are performed for time T, which we will choose sufficiently large so that particles have the time to travel through X and be measured.

2.2 Semigroups and unbounded operators

We introduce the following space

$$\mathcal{Z} := \{f \in L^1(X \times \mathbb{S}^{n-1}) \mid v \cdot \nabla_x f \in L^1(X \times \mathbb{S}^{n-1})\}, \tag{2.3}$$

$$\|f\|_{\mathcal{Z}} := \|f\|_{L^1(X\times\mathbb{S}^{n-1})} + \|v \cdot \nabla_x f\|_{L^1(X\times\mathbb{S}^{n-1})}; \tag{2.4}$$

where $v \cdot \nabla_x$ is understood in the distributional sense.

It is known (see [7, 8]) that the trace map γ_- from $C^1(\bar{X} \times \mathbb{S}^{n-1})$ to $C(\Gamma_-)$ defined by

$$\gamma_-(f) = f_{|\Gamma_-} \tag{2.5}$$

extends to a continuous operator from $\mathcal{Z}$ onto $L^1(\Gamma_-, \tau_+(x, v)d\xi(x, v))$ and admits a continuous lifting. Note that $L^1(\Gamma_-, d\xi)$ is a subset of the spaces $L^1(\Gamma_-, \tau_+(x, v)d\xi(x, v))$.

We introduce the following notation

$$A_1 f = -\sigma f,\ A_2 f = \int_{\mathbb{S}^{n-1}} k(x, v', v)f(x, v')dv'. \tag{2.6}$$

As (σ, k) is admissible, the operators A_1 and A_2 are bounded operators in $L^1(X \times \mathbb{S}^{n-1})$.

Consider the following unbounded operators

$$T_1 f = -v \cdot \nabla_x f + A_1 f,\ D(T_1) = \{f \in \mathcal{Z} \mid f_{|\Gamma_-} = 0\}, \tag{2.7}$$

$$T f = T_1 f + A_2 f,\ D(T) = D(T_1). \tag{2.8}$$

It is known that the unbounded operators T_1 and T are generators of strongly continuous semigroups in $L^1(X \times \mathbb{S}^{n-1})$ $U_1(t)$, $U(t)$ respectively (see for example [11, Proposition 2 pp

226]). In addition $U_1(t)$ and $U(t)$ preserve the cone of positive functions, and $U_1(t)$ is given explicitly by the following formula

$$U_1(t)f = e^{-\int_0^t \sigma(x-sv,v)ds} f(x-tv,v)\Theta(x,x-tv),\ \text{for a.e. } (x,v) \in X \times \mathbb{S}^{n-1}, \tag{2.9}$$

for $f \in L^1(X \times \mathbb{S}^{n-1})$, where

$$\Theta(x,y) = \begin{cases} 1 \text{ if } x + p(y-x) \in X \text{ for all } p \in (0,1], \\ \qquad\qquad 0 \text{ otherwise,} \end{cases} \tag{2.10}$$

for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

We recall the Dyson-Phillips formula

$$U(t) = \sum_{m=0}^{+\infty} H_m(t) \tag{2.11}$$

for $t \geq 0$, where

$$H_m(t) := \int\limits_{\substack{s_1\geq 0,\ldots,s_m\geq 0\\ s_1+\ldots+s_m\leq t}} U_1(t-s_1-\ldots-s_m)A_2U_1(s_1)\ldots A_2U_1(s_m)ds_1\ldots ds_m,\ m\geq 1, \tag{2.12}$$

$$H_m(t) = \int_0^t H_{m-1}(t-s)A_2U_1(s)ds,\ m \geq 1, \tag{2.13}$$

$$H_0(t) := U_1(t). \tag{2.14}$$

2.3 Trace results

We introduce the following space

$$\mathcal{W} := \{u \in L^1((0,T)\times X \times \mathbb{S}^{n-1}) \mid \left(\frac{\partial}{\partial t} + v\cdot\nabla_x\right) u \in L^1((0,T)\times X\times \mathbb{S}^{n-1})\}, \tag{2.15}$$

$$\|u\|_{\mathcal{W}} := \|u\|_{L^1((0,T)\times X\times\mathbb{S}^{n-1})} + \left\| \left(\frac{\partial}{\partial t} + v\cdot\nabla_x\right) u \right\|_{L^1((0,T)\times X\times\mathbb{S}^{n-1})}; \tag{2.16}$$

where $\frac{\partial}{\partial t}$ and $v \cdot \nabla_x$ are understood in the distributional sense.

It is known (see [7, 8]) that the trace map γ_- (respectively γ_+) from $C^1([0,T]\times\bar{X}\times\mathbb{S}^{n-1})$ to $C(X\times\mathbb{S}^{n-1})\times C((0,T)\times\Gamma_\pm)$ defined by

$$\gamma_-(\psi) = (\psi(0,.), \psi_{|(0,T)\times\Gamma_-}) \text{ (respectively } \gamma_+(\psi) = (\psi(T,.), \psi_{|(0,T)\times\Gamma_+}) \tag{2.17}$$

extends to a continuous operator from $\mathcal{W}$ onto $L^1(X\times\mathbb{S}^{n-1}, \tau_+(x,v)dxdv) \times L^1((0,T)\times\Gamma_-, \min(T-t,\tau_+(x,v))dtd\xi(x,v))$ (respectively $L^1(X\times\mathbb{S}^{n-1},\tau_-(x,v)dxdv)\times L^1((0,T)\times\Gamma_+, \min(t,\tau_-(x,v))dtd\xi(x,v)))$. In addition $\gamma_\pm$ admits a continuous lifting. Note that $L^1(X\times\mathbb{S}^{n-1})$ is a subset of $L^1(X\times\mathbb{S}^{n-1},\tau_+(x,v)dxdv)$. Note also that $L^1((0,T)\times\Gamma_-,dtd\xi)$ (resp. $L^1((0,T)\times\Gamma_+,dtd\xi)$) is a subset of $L^1((0,T)\times\Gamma_-,\min(T-t,\tau_+(x,v))dtd\xi(x,v))$ (resp. $L^1((0,T)\times\Gamma_+,\min(t, \tau_-(x,v))dtd\xi(x,v)))$.

We now introduce the space

$$\tilde{\mathcal{W}} := \{u \in \mathcal{W} \mid \gamma_-(u) \in L^1(X\times\mathbb{S}^{n-1})\times L^1((0,T)\times\Gamma_-,dtd\xi)\}. \tag{2.18}$$

We recall the following trace results (owed to [7, 8] in a more general setting).

Lemma 2.1. *The following equality is valid*

$$\tilde{\mathcal{W}} = \{u \in \mathcal{W} \mid \gamma_+(u) \in L^1(X \times \mathbb{S}^{n-1}) \times L^1((0,T) \times \Gamma_+, dtd\xi)\}. \tag{2.19}$$

In addition the trace maps

$$\gamma_\pm : \tilde{\mathcal{W}} \to L^1(X\times\mathbb{S}^{n-1})\times L^1((0,T)\times\Gamma_\pm, dtd\xi) \text{ are continuous, onto, and admit continuous lifting.} \tag{2.20}$$

2.4 Solution to equation (2.2)

We identify the space $L^1((0,r), L^1(\Gamma_\pm, d\xi))$ with the space $L^1((0,r) \times \Gamma_\pm, dtd\xi)$ for any $r > 0$. We extend by 0 on $\mathbb{R}$ outside the interval $(0,\eta)$ any function $\phi \in L^1((0,\eta), L^1(\Gamma_-, d\xi))$.

Let $\phi \in L^1((0,\eta), L^1(\Gamma_-, d\xi))$. Then we consider the lifting $G_-(t)\phi \in \tilde{\mathcal{W}}$ of $(0,\phi)$ defined by

$$G_-(t)\phi(x,v) := e^{-\int_0^{\tau_-(x,v)}\sigma(x-sv,v)ds}\phi_-(t-\tau_-(x,v), x-\tau_-(x,v)v, v), \text{ for a.e. } (t,x,v) \in (0,T)\times X\times\mathbb{S}^{n-1}. \tag{2.21}$$

Note that $G_-(.)\phi$ is a solution in the distributional sense of the equation $(\frac{\partial}{\partial t} + v\cdot\nabla_x)u + \sigma u = 0$ in $(0,T) \times X \times \mathbb{S}^{n-1}$ and

$$\|G_-(.)\phi\|_{\mathcal{W}} \le (1 + \|\sigma\|_\infty)\|G_-(.)\phi\|_{L^1((0,T)\times X\times\mathbb{S}^{n-1})} \le (1 + \|\sigma\|_\infty)T\|\phi_-\|_{L^1((0,\eta)\times\Gamma_-,dtd\xi)}. \tag{2.22}$$

To prove this two latter statements, one can use the following change of variables (see [10]).

Lemma 2.2. *We have*

$$\int_{X\times\mathbb{S}^{n-1}} f(x,v)dxdv = \int_{\Gamma_\mp}\int_0^{\tau_\pm(x,v)} f(x \pm tv)dtd\xi(x,v), \tag{2.23}$$

for $f \in L^1(X \times V)$.

From (2.22) we obtain that the map $i : L^1((0,\eta), L^1(\Gamma_-, d\xi)) \to \tilde{\mathcal{W}}$ defined by

$$i(\phi) = G_-(.)\phi,\ \ \phi \in L^1((0,\eta), L^1(\Gamma_-, d\xi)), \tag{2.24}$$

is continuous.

The following result holds (see [11, Theorem 3 p. 229]).

Lemma 2.3. *The equation* (2.2) *admits a unique solution u in $\tilde{\mathcal{W}}$ which is given by*

$$u(t) = G_-(t)\phi + \int_0^t U(t-s)A_2G_-(s)\phi ds. \tag{2.25}$$

where $U(t)$ is the strongly continuous semigroup in $L^1(X \times \mathbb{S}^{n-1})$ introduced in subsection 2.2.

Using (2.25) and the Dyson-Phillips expansion (2.11) we obtain that the solution u of (2.2) may be decomposed as

$$u(t) = G_-(t)\phi + \sum_{m=0}^{\infty}\int_{-\infty}^t H_m(t-s)A_2G_-(s)\phi ds, \tag{2.26}$$

for $t \ge 0$ and $\phi \in L^1((0,\eta) \times \partial X, dtd\mu(x))$. The first term in the above series $G_-(t)\phi$ is the ballistic part of $u(t)$ while the term corresponding to $m \ge 1$ is m-linear in the scattering kernel k. The term corresponding to $m = 1$ is the single scattering term.

From (2.24), Lemma 2.3 and (2.20), we also obtain the existence of the albedo operator.

Lemma 2.4. *The albedo operator A given by the formula*

$$A\phi = u_{|(0,T)\times\Gamma_+},\ \text{for } \phi \in L^1((0,\eta), L^1(\Gamma_-, d\xi)) \text{ where } u \text{ is given by (2.25)}, \tag{2.27}$$

is well-defined and is a bounded operator from $L^1((0,\eta), L^1(\Gamma_-, d\xi))$ to $L^1((0,T), L^1(\Gamma_+, d\xi))$.

We refer the reader to [9] for the reconstruction of the optical parameters when the full albedo operator is known. We assume here that only partial knowledge of the albedo operator is available from measurements.

3 The operator $A_{S,W}$ and its distributional kernel

3.1 Angularly averaged measurements

We now define more precisely the type of measurements we consider in this paper. The directional behavior of the source term is determined by a fixed function $S(x,v)$, which is bounded and continuous on Γ_-. We assume that the incoming conditions have the following structure

$$\phi_S(t', x', v') = S(x', v')\phi(t', x'),\ t' \in (0,\eta),\ (x', v') \in \Gamma_-, \tag{3.1}$$

where $\phi(t,x)$ is an arbitrary function in $L^1((0,\eta)\times\partial X)$. We model the detectors by the kernel $W(x,v)$, which we assume is a continuous and bounded function on Γ_+. The available measurements are therefore modeled by the availability of the averaged albedo operator $A_{S,W}$ from $L^1((0,\eta)\times\partial X, dtd\mu(x))$ to $L^1((0,T)\times\partial X, dtd\mu(x))$ and defined by

$$A_{S,W}\phi(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}} A(\phi_S)(t,x,v)W(x,v)(\nu(x)\cdot v)dv,\ \text{for a.e. } (t,x) \in (0,T)\times\partial X. \tag{3.2}$$

The functions S and W are fixed throughout the paper. The case $W \equiv 1$ corresponds to measurements of the current of exiting particles at the domain's boundary.

The decomposition of the transport solution (2.26) translates into a similar decomposition of the albedo operator of the form

$$A_{S,W}\phi(t,x) = \sum_{m=0}^{+\infty} A_{m,S,W}\phi(t,x), \tag{3.3}$$

for $(t,x) \in (0,T)\times\partial X$, where we have defined

$$A_{0,S,W}\phi(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x)\cdot v)W(x,v)\left(G_-(.)\phi_S\right)_{|(0,T)\times\Gamma_+}(t,x,v)dv, \tag{3.4}$$

$$A_{m,S,W}\phi(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x)\cdot v)W(x,v)\left(\int_{-\infty}^{t} H_{m-1}(t-s)A_2G_-(s)\phi_S ds\right)_{|(0,T)\times\Gamma_+}(t,x,v)dv, \tag{3.5}$$

for a.e. $(t,x) \in (0,T)\times\partial X$ where ϕ_S is defined by (3.1). The kernels of the operators $A_{m,S,W}$ can be written explicitly.

3.2 Distributional kernel of the operators $A_{m,S,W}$

Consider the nonnegative measurable E from $\partial X\times\partial X\to\mathbb{R}$ defined by

$$E(x_1,x_2)=\begin{cases} e^{-\int_0^{|x_1-x_2|}\sigma(x_1-s\frac{x_1-x_2}{|x_1-x_2|},\frac{x_1-x_2}{|x_1-x_2|})ds} & \text{if } x_1+p(x_2-x_1)\in X \text{ for all } p\in(0,1),\\ 0 \text{ otherwise,} & \end{cases} \tag{3.6}$$

for a.e. $(x_1,x_2)\in\partial X\times\partial X$. For $m\ge 3$, we also define the nonnegative measurable real function $E(x_1,\ldots,x_m)$ by the formula

$$E(x_1,\ldots,x_m)=e^{-\sum_{i=1}^{m-1}\int_0^{|x_i-x_{i+1}|}\sigma(x_i-s\frac{x_i-x_{i+1}}{|x_i-x_{i+1}|},\frac{x_i-x_{i+1}}{|x_i-x_{i+1}|})ds}\Theta(x_m,x_{m-1})\Pi_{i=1}^{m-2}\Theta(x_i,x_{i+1}), \tag{3.7}$$

for a.e. $(x_1,\ldots,x_m)\in\partial X\times(\mathbb{R}^n)^{m-2}\times\partial X$, where Θ is defined by (2.10). The function $E(x_1,\ldots,x_m)$ measures the total attenuation along the broken path $(x_1,\ldots,x_m)\in\partial X\times\mathbb{R}^{m-2}\times\partial X$ provided $(x_2,\ldots,x_{m-1})\in X^{m-2}$.

For $m\in\mathbb{N}$, $m\ge 1$ and for any subset U of $\mathbb{R}^m$ we denote by χ_U the characteristic function from $\mathbb{R}^m$ to $\mathbb{R}$ defined by $\chi_U(y)=1$ when $y\in U$ and $\chi_U(y)=0$ otherwise. Using (3.4)–(3.5), (2.21) and (2.13)–(2.14) we then obtain the following result on the structure of the kernels of the albedo operator.

Proposition 3.1. *We have*

$$A_{m,S,W}(\phi)(t,x)=\int_{(0,\eta)\times\partial X}\gamma_m(t-t',x,x')\phi(t',x')dt'd\mu(x'), \tag{3.8}$$

for $m\ge 0$ and for a.e. $(t,x)\in(0,T)\times\partial X$, where

$$\gamma_0(\tau,x,x') := \frac{E(x,x')}{|x-x'|^{n-1}}\left[W(x,v)S(x',v)(\nu(x)\cdot v)|\nu(x')\cdot v|\right]_{v=\frac{x-x'}{|x-x'|}}\delta(\tau-|x-x'|), \tag{3.9}$$

$$\begin{aligned}\gamma_1(\tau,x,x') &:= \chi_{(0,+\infty)}(\tau-|x'-x|)\int_{\mathbb{S}^{n-1}_{x,+}}(\nu(x)\cdot v)W(x,v)\,[E(x,x-sv,x')k(x-sv,v',v)\\ &\times \chi_{(0,\tau_-(x,v))}(s)S(x',v')|\nu(x')\cdot v'|]\big|_{v'=\frac{x-x'-sv}{|x-x'-sv|}\,;\,s=\frac{\tau^2-|x-x'|^2}{2(\tau-v\cdot(x-x'))}}\frac{2^{n-2}(\tau-(x-x')\cdot v)^{n-3}}{|x-x'-\tau v|^{2n-4}}dv,\end{aligned} \tag{3.10}$$

for $(\tau,x,x')\in\mathbb{R}\times\partial X\times\partial X$ and where γ_m for $m\ge 2$ admits a similar, more complex, expression given in Section 8 (see (8.12)–(8.13)*).*

Because the above formulas are central in our uniqueness and stability results, we briefly present their derivation and refer the reader to Section 10 for the rest of the proof of Proposition 3.1.

Derivation of (3.9) *and* (3.10). From (3.4) and the definition of G_-, we obtain $A_{0,S,W}\phi(t,x)=\int_{\mathbb{S}^{n-1}_{x,+}}(\nu(x)\cdot v)W(x,v)E(x,x-\tau_-(x,v)v)S(x-\tau_-(x,v)v,v)\phi(t-\tau_-(x,v),x-\tau_-(x,v)v)dv$, $(t,x)\in(0,T)\times\partial X$ and for $\phi\in L^1((0,\eta)\times\partial X)$. Therefore, performing the change of variables "x'"$=x-\tau(x,v)v$ ($dv=\frac{|\nu(x')\cdot v|}{|x-x'|^{n-1}}d\mu(x')$ and $\tau(x,v)=|x-x'|$), we obtain (3.9).

From the definition of A_2 and G_- we note that $A_2G_-(s)\phi_S(z,w):=\int_{\mathbb{S}^{n-1}}k(z,v',w)E(z,z-\tau_-(z,v')v')S(z-\tau_-(z,v')v',v')\phi(s-\tau_-(z,v'),z-\tau_-(z,v')v')dv'$, for a.e. $(z,w)\in X\times\mathbb{S}^{n-1}$ and for $\phi\in L^1((0,\eta)\times\partial X)$. Performing the change of variables "$x'=z-\tau_-(z,v')v'$", we obtain the equality $(A_2G_-(s)\phi_S)(z,w)=\int_{\partial X}\left[k(z,v',w)S(x',v')|\nu(x')\cdot v'|\right]_{v'=\frac{z-x'}{|z-x'|}}\frac{E(z,x')}{|z-x'|^{n-1}}\phi(s-|z-x'|,x')d\mu(x')$, for a.e. $(z,w)\in X\times\mathbb{S}^{n-1}$ and $\phi\in L^1((0,\eta)\times\partial X)$. Using also the definition of

$A_{1,S,W}$ (see (3.5) for $m=1$) we obtain the following equality for any $\phi \in L^1((0,\eta)\times\partial X)$ and for a.e. $(t,x)\in(0,T)\times\partial X$

$$A_{1,S,W}(\phi)(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}} \int_{-\infty}^{t} \int_{\partial X} \left[k(x-(t-s)v,v',v)S(x',v')|\nu(x')\cdot v'|\right]_{v'=\frac{x-(t-s)v-x'}{|x-(t-s)v-x'|}} (\nu(x)\cdot v)$$
$$\times \frac{E(x,x-(t-s)v,x')}{|x-(t-s)v-x'|^{n-1}} \Theta(x,x-(t-s)v)\phi(s-|x-(t-s)v-x'|,x')W(x,v)d\mu(x')dsdv. \quad (3.11)$$

Then performing the changes of variables "s"$=t-s$ and "t'"$=t-s-|x-sv-x'|$ ($s=\frac{(t-t')^2-|x-x'|^2}{2(t-t'-v\cdot(x-x'))}$, $\frac{dt'}{ds}=\frac{2((t-t')-(x-x')\cdot v)^2}{|x-x'-(t-t')v|^2}$), we obtain (3.10). $\square$

To simplify notation, we define the multiple scattering kernels

$$\Gamma_k = \sum_{m=k}^{\infty} \gamma_m. \quad (3.12)$$

3.3 Regularity of the albedo kernels

The reconstruction of the optical parameters is based on an analysis of the behavior in time of the kernels of the albedo operator. Our first result in this direction is the following.

Theorem 3.2. *Assume that $k\in L^\infty(X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$. Then the following holds:*

$$\sqrt{\tau^2-|x-x'|^2}\gamma_1(\tau,x,x') \in L^\infty((0,T)\times\partial X\times\partial X) \qquad \text{when } n=2; \quad (3.13)$$
$$\frac{\tau|x-x'|}{\ln\left(\frac{\tau+|x-x'|}{\tau-|x-x'|}\right)}\gamma_1(\tau,x,x') \in L^\infty((0,T)\times\partial X\times\partial X) \qquad \text{when } n=3; \quad (3.14)$$
$$\tau|x-x'|^{n-2}\gamma_1(\tau,x,x') \in L^\infty((0,T)\times\partial X\times\partial X) \qquad \text{when } n\geq 4. \quad (3.15)$$

In addition, assume that $k\in L^\infty(X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ and that there exists $\delta>0$ such that $\mathrm{supp}k\subseteq\{y\in X \mid \inf_{x\in\partial X}|x-y|\geq\delta\}$, *i.e., the scattering coefficient vanishes in the vicinity of ∂X. Then, the following holds*

$$(\tau-|x-x'|)^{\frac{3-n}{2}}\gamma_1(\tau,x,x') \in L^\infty((0,T)\times\partial X\times\partial X) \qquad \text{when } n\geq 2. \quad (3.16)$$

Theorem 3.2 is proved in Section 7. The results (3.14) and (3.15) of Theorem 3.2 correspond to singularities of the single scattering contribution that depend on the values of k on ∂X. The above theorem shows that the structure of the single scattering coefficient is quite different depending on whether k vanishes on ∂X or not.

The following result describes some regularity properties of the multiple scattering. It is because multiple scattering is *more regular* than single scattering, in an appropriate sense, that we can reconstruct the scattering coefficient in a stable manner.

Theorem 3.3. *Assume that $k\in L^\infty(X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$. Then the following holds:*

$$\Gamma_2(\tau,x,x') \in L^\infty((0,T)\times\partial X\times\partial X), \qquad \text{when } n=2; \quad (3.17)$$
$$\frac{\tau|x-x'|\Gamma_2(\tau,x,x')}{(\tau-|x-x'|)\left(1+\ln\left(\frac{\tau+|x-x'|}{\tau-|x-x'|}\right)\right)^2} \in L^\infty((0,T)\times\partial X\times\partial X), \qquad \text{when } n=3; \quad (3.18)$$
$$\frac{\tau|x-x'|^{n-2}}{\tau-|x-x'|}\Gamma_2(\tau,x,x') \in L^\infty((0,T)\times\partial X\times\partial X), \qquad \text{when } n\geq 4. \quad (3.19)$$

In addition, assume that $k \in L^\infty(X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ *and that there exists* $\delta > 0$ *such that* $\mathrm{supp} k \subseteq \{y \in X \mid \inf_{x\in\partial X} |x-y| \ge \delta\}$. *Then the following holds:*

$$(\tau - |x-x'|)^{-1}\left(1+\ln\left(\frac{\tau+|x-x'|}{\tau-|x-x'|}\right)\right)^{-1}\Gamma_2(\tau,x,x') \in L^\infty((0,T)\times\partial X\times\partial X), \quad \textit{when } n=3; \tag{3.20}$$

$$(\tau-|x-x'|)^{\frac{1-n}{2}}\Gamma_2(\tau,x,x') \in L^\infty((0,T)\times\partial X\times\partial X), \quad \textit{when } n\ge 4. \tag{3.21}$$

Theorem 3.3 is proved in Section 8. These results quantify how "smoother" multiple scattering is compared to the single scattering contribution considered in Theorem 3.2.

3.4 Asymptotics of the single scattering term

In this subsection we assume that X is also convex. We give limits for the single scattering term in two configurations given by:

$$\begin{array}{l}\text{the nonnegative function } \sigma \text{ is bounded and continuous on } X \times \mathbb{S}^{n-1},\\ \text{the nonnegative function } k \text{ is continuous on } \bar{X}\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1};\end{array} \tag{3.22}$$

or

$$\begin{array}{l}\text{there exists a convex open subset } Y \subseteq X \text{ with } C^1 \text{ boundary such that } \sigma(x,v)=0\\ \text{for } (x,v)\in(X\backslash Y)\times\mathbb{S}^{n-1} \text{ and the nonnegative function } \sigma \text{ is bounded and}\\ \text{continuous on } Y\times\mathbb{S}^{n-1}\text{; and there exists a convex open subset } Z\subseteq Y\subseteq X\\ \text{with } C^1 \text{ boundary such that } \delta := \inf_{(x,z)\in\partial X\times Z}|x-z|>0 \text{ and } k(x,v',v)=0\\ \text{for } (x,v',v)\in(X\backslash Z)\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}, \text{ and the nonnegative function } k\\ \text{is bounded and continuous on } Z\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1},\end{array} \tag{3.23}$$

When either (3.22) or (3.23) is satisfied, we want to analyze the behavior of the function $\gamma_1(\tau,x,x')$ given by the right hand side of (3.10) for all $(\tau,x,x')\in\mathbb{R}\times\partial X\times\partial X$. We need to introduce some notation. Let $\vartheta_0 : \mathbb{S}^{n-1}\times X \to \mathbb{R}$ be the function defined by

$$\vartheta_0(v,x) = (\tau_-(x,v)\tau_+(x,v))^{-\frac{n-1}{2}}, \ (v,x)\in\mathbb{S}^{n-1}\times X, \tag{3.24}$$

and consider the weighted X-ray transform P_{ϑ_0} defined by

$$P_{\vartheta_0}f(v,x) = \int_{\tau_-(x,v)}^{\tau_+(x,v)} \vartheta_0(v,tv+x)f(tv+x)dt, \tag{3.25}$$

for a.e. $(v,x)\in\mathbb{S}^{n-1}\times\partial X$ and $f\in L^2(X,\sup_{v\in\mathbb{S}^{n-1}}\vartheta_0(v,x)dx)$. The first result analyzes the behavior of γ_1 under hypothesis (3.22).

Theorem 3.4. *Assume that the open subset* X *of* $\mathbb{R}^n$ *with* C^1 *boundary is convex. Let* $(x,x_0')\in\partial X^2$ *be such that* $x+s(x-x_0')\in X$ *for some* $s\in(0,1)$. *Set* $v_0=\frac{x-x_0'}{|x-x_0'|}$ *and* $t_0=|x-x_0'|$. *Then under condition* (3.22), *we have the following results. When* $n=2$, *then*

$$\begin{aligned}\gamma_1(\tau,x,x_0') = \ & \frac{1}{\sqrt{\tau-t_0}}\frac{\sqrt{2}W(x,v_0)S(x_0',v_0)(\nu(x)\cdot v_0)|\nu(x_0')\cdot v_0|E(x,x_0')}{\sqrt{t_0}}\\ & \times P_{\vartheta_0}k_{v_0}(v_0,x)+o\Big(\frac{1}{\sqrt{\tau-t_0}}\Big), \quad \textit{as } \tau\to t_0^+,\end{aligned} \tag{3.26}$$

where P_{ϑ_0} is defined by (3.25) *and $k_{v_0}(y) := k(y, v_0, v_0)$ for $y \in X$.*
When $n = 3$, then

$$\begin{aligned}\gamma_1(\tau, x, x_0') &= \ln(\frac{1}{\tau - t_0})\frac{\pi}{t_0^2} W(x, v_0)S(x_0', v_0)(\nu(x)\cdot v_0)|\nu(x_0')\cdot v_0|E(x, x_0')\\ &\quad \times (k(x, v_0, v_0) + k(x_0', v_0, v_0)) + o\Big(\ln(\frac{1}{\tau - t_0})\Big), \text{ as } \tau \to t_0^+.\end{aligned} \tag{3.27}$$

When $n \geq 4$, then

$$\begin{aligned}\gamma_1(\tau, x, x_0') &= t_0^{1-n}E(x, x_0')\Bigg[S(x_0', v_0)|\nu(x_0')\cdot v_0| \int_{\mathbb{S}^{n-1}_{x,+}} \frac{W(x, v)(\nu(x)\cdot v)k(x, v_0, v)}{1 - v\cdot v_0}dv\\ &\quad + W(x, v_0)(\nu(x)\cdot v_0)\int_{\mathbb{S}^{n-1}_{x_0',-}} \frac{k(x_0', v', v_0)S(x_0', v')|\nu(x_0')\cdot v'|}{1 - v'\cdot v_0}dv'\Bigg]\\ &\quad + o(1), \text{ as } \tau \to t_0^+.\end{aligned} \tag{3.28}$$

Theorem 3.4 is proved in Section 5. Note that γ_1 depends on the value of k on ∂X in dimension $n \geq 3$. Under hypothesis (3.23), i.e., when the scattering coefficient vanishes in the vicinity of where measurements are collected, we have the quite different behavior:

Theorem 3.5. *Assume that the open subset X of $\mathbb{R}^n$ with C^1 boundary is also convex and assume that condition* (3.23) *is fulfilled. Let $(x, x_0') \in \partial X^2$ be such that $x_0' + s(x - x_0') \in Z$ for some $s \in (0, 1)$. Set $v_0 = \frac{x - x_0'}{|x - x_0'|}$ and $t_0 = |x - x_0'|$. Then we have the following.*

When $n = 2$, then (3.26) *still holds.*

When $n \geq 3$, then

$$\begin{aligned}&\gamma_1(\tau, x, x_0') = (\tau - t_0)^{\frac{n-3}{2}}(2t_0)^{\frac{1-n}{2}}\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})S(x_0', v_0)W(x, v_0)|\nu(x_0')\cdot v_0|(\nu(x)\cdot v_0)\\ &\times E(x, x_0')P_{\vartheta_0}k_{v_0}(v_0, x) + o((\tau - t_0)^{\frac{n-3}{2}}), \text{ as } \tau \to t_0^+,\end{aligned} \tag{3.29}$$

where P_{ϑ_0} is defined by (3.25) *and $k_{v_0}(y) := k(y, v_0, v_0)$ for $y \in X$.*

Theorem 3.5 is proved in Section 5. Theorem 3.5 may remain valid under different conditions from those stated in (3.23). For instance, when σ is bounded and continuous on X and k is continuous on $X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ and $k(x, ., .)$ decays sufficiently rapidly as x get closer and closer to the boundary ∂X for any $x \in X$, then the same asymptotics of γ_1 holds.

4 Uniqueness and stability results

We denote by $\gamma := \Gamma_0 = \sum_{m=0}^{+\infty}\gamma_m$ the distributional kernel of $A_{S,W}$. Then $\gamma - \gamma_0 = \Gamma_1$ denotes the distributional kernel of the multiple scattering of $A_{S,W}$. For the rest of the paper, we assume that the duration of measurement $T > \mathrm{diam}(X) := \sup_{(x,y)\in X^2}|x - y|$ so that the singularities of the ballistic and single scattering contributions are indeed captured by the available measurements.

Let $(\tilde\sigma, \tilde k)$ be a pair of absorption and scattering coefficients that also satisfy (2.1). We denote by a superscript ˜ any object (such as the albedo operator $\tilde A$ or the distributional kernels $\tilde\gamma$ and $\tilde\gamma_0$) associated to $(\tilde\sigma, \tilde k)$. Moreover if (σ, k) satisfies (3.23) for some (Y, Z) and $(\tilde\sigma, \tilde k)$ also satisfies (3.23) for some $(\tilde Y, \tilde Z)$, then we always make the additional assumption $Y = \tilde Y$ and $Z = \tilde Z$. Let $\|.\|_{\eta,T} := \|.\|_{\mathcal{L}(L^1((0,\eta)\times\partial X)), L^1((0,T)\times\partial X))}$.

4.1 Stability estimates under condition (3.22) or (3.23)

Theorem 4.1. *Assume that the open subset X of $\mathbb{R}^n$ with C^1 boundary is also convex. Let (σ, k) and $(\tilde{\sigma}, \tilde{k})$ satisfy condition (3.22). Let $x'_0 \in \partial X$. Then we have:*

$$\int_{\partial X} \left[\frac{|E - \tilde{E}|(x, x'_0)}{|x - x'_0|^{n-1}} W(x, v_0) S(x', v_0)(\nu(x) \cdot v_0)|\nu(x'_0) \cdot v_0| \right]_{\substack{t_0 = |x - x'_0| \\ v_0 = \frac{x - x'_0}{|x - x'_0|}}} d\mu(x) \le \|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T}. \tag{4.1}$$

Let $x \in \partial X$ be such that $px'_0 + (1-p)x \in X$ for some $p \in (0,1)$. Set $v_0 = \frac{x - x'_0}{|x - x'_0|}$ and $t_0 = |x - x'_0|$. When $n = 2$, we have

$$W(x, v_0) S(x'_0, v_0)(\nu(x) \cdot v_0)|\nu(x'_0) \cdot v_0| \left| E(x, x'_0) P_{\vartheta_0} k_{v_0}(v_0, x) - \tilde{E}(x, x'_0) P_{\vartheta_0} \tilde{k}_{v_0}(v_0, x) \right|$$
$$\le \frac{1}{2} \left\| \sqrt{\tau^2 - |z - z'|^2} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \right\|_{L^\infty}, \tag{4.2}$$

where $\|\cdot\|_{L^\infty} := \|\cdot\|_{L^\infty((0,T)\times\partial X\times\partial X)}$, P_{ϑ_0} is defined by (3.25) and $k_{v_0}(y) := k(y, v_0, v_0)$ for $y \in X$ ($\tilde{k}_{v_0}$ is defined similarly).

When $n = 3$, then

$$\left| E(x, x'_0)(k(x, v_0, v_0) + k(x'_0, v_0, v_0)) - \tilde{E}(x, x'_0)(\tilde{k}(x, v_0, v_0) + \tilde{k}(x'_0, v_0, v_0)) \right|$$
$$\times W(x, v_0) S(x'_0, v_0)(\nu(x) \cdot v_0)|\nu(x'_0) \cdot v_0| \le \frac{1}{\pi} \left\| \frac{\tau|z - z'|}{\ln\left(\frac{\tau + |z - z'|}{\tau - |z - z'|}\right)} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \right\|_{L^\infty}. \tag{4.3}$$

When $n \ge 4$, then

$$S(x'_0, v_0)|\nu(x'_0) \cdot v_0| \Bigg| \int_{\mathbb{S}^{n-1}_{x,+}} \frac{W(x, v)(\nu(x) \cdot v)}{1 - v \cdot v_0} \Big(E(x, x'_0) k(x, v_0, v) - \tilde{E}(x, x'_0) \tilde{k}(x, v_0, v) \Big) dv$$
$$+ W(x, v_0)(\nu(x) \cdot v_0) \int_{\mathbb{S}^{n-1}_{x'_0,-}} \frac{S(x'_0, v')|\nu(x'_0) \cdot v'|}{1 - v' \cdot v_0} \Big(E(x, x'_0) k(x'_0, v', v_0) - \tilde{E}(x, x'_0) \tilde{k}(x'_0, v', v_0) \Big) dv' \Bigg|$$
$$\le \left\| \tau |z - z'|^{n-2} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \right\|_{L^\infty}. \tag{4.4}$$

Theorem 4.1 is proved in Section 7. It shows that the spatial structure of k may be stably reconstructed at the domain's boundary. More interesting is the following theorem, which provides some stability of the reconstruction of the scattering coefficient when it vanishes in the vicinity of the boundary ∂X.

Theorem 4.2. *Assume that the open subset X of $\mathbb{R}^n$ with C^1 boundary is also convex. Assume also that $\inf_{(x',v')\in\Gamma_-} S(x', v') > 0$ and $\inf_{(x,v)\in\Gamma_+} W(x, v) > 0$. Let (σ, k) and $(\tilde{\sigma}, \tilde{k})$ satisfy condition (3.23). Let $x'_0 \in \partial X$. Then there exist constants $C_1 = C_1(S, W, X, Y)$ and $C_2 = C_2(S, W, X, Z)$ such that*

$$\int_{\mathbb{S}^{n-1}_{x'_0,-}} |E - \tilde{E}|(x'_0 + \tau_+(x'_0, v_0)v_0, x'_0)|\nu(x'_0) \cdot v_0| dv_0 \le C_1 \|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T}, \tag{4.5}$$

$$\left| E(x, x'_0) P_{\vartheta_0} k_{v'_0}(v'_0, x'_0) - \tilde{E}(x, x'_0) P_{\vartheta_0} \tilde{k}_{v'_0}(v'_0, x'_0) \right| \le C_2 \left\| (\tau - |z - z'|)^{\frac{3-n}{2}} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \right\|_{L^\infty}, \tag{4.6}$$

for $x \in \partial X$ such that $px_0' + (1-p)x \in Z$ for some $p \in (0,1)$ where $v_0' = \frac{x-x_0'}{|x-x_0'|}$, P_{ϑ_0} is defined in (3.25)*, and $k_{v_0'}(y) := k(y, v_0', v_0')$ for $y \in X$ ($\tilde{k}_{v_0'}$ is defined similarly).*

Theorem 4.2, which is one of the main results of this paper, is proved in Section 8.

4.2 The case when X is a ball of $\mathbb{R}^n$

When X is an open Euclidean ball of $\mathbb{R}^n$, which is important from the practical point of view in medical imaging as it is relatively straightforward to place sources and detectors on a sphere, we are able to invert the weighted X-ray transform $P_{\vartheta_0}f$, $f(x,v) := f(x) \in L^2(X, \sup_{v\in\mathbb{S}^{n-1}} \vartheta_0(v,x)dx)$ using the classical inverse X-ray transform (inverse Radon transform in dimension $n = 2$). In the next subsection we shall consider a larger class of domains X, which requires one to solve more complex weighted X-ray transforms.

Up to rescaling, we assume $X = B_n(0,1)$, the ball in $\mathbb{R}^n$ centered at 0 of radius 1. Consider the X-ray transform P defined by

$$Pf(v,x) = \int_{\tau_-(x,v)}^{\tau_+(x,v)} f(sv+x)ds \text{ for a.e. } (v,x) \in \mathbb{S}^{n-1} \times \partial X, \tag{4.7}$$

for $f \in L^2(X)$ (we extend f by 0 outside X). We have the following Proposition 4.3.

Proposition 4.3. *When $X = B_n(0,1)$ we have*

$$P_{\vartheta_0} f(v,x) = P(\varrho f)(v,x), \text{ for a.e. } (v,x) \in \mathbb{S}^{n-1} \times \partial X, \tag{4.8}$$

for $f \in L^2(X, \sup_{v\in\mathbb{S}^{n-1}} \vartheta_0(v,x)dx)$ where $\varrho(y) := (1-|y|^2)^{-\frac{n-1}{2}}$, $y \in X$.

Proof of Proposition 4.3. It is easy to see that

$$\begin{aligned} \tau_\pm(tv + qv^\perp, v) &= \sqrt{1-q^2} \mp t, & (4.9)\\ \vartheta_0(v,x) &= (1-q^2-t^2)^{-\frac{n-1}{2}} = (1-|x|^2)^{-\frac{n-1}{2}}, & (4.10) \end{aligned}$$

for $(t,q) \in \mathbb{R}^2$, $t^2 + q^2 \le 1$ and for $(v, v^\perp) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, $v \cdot v^\perp = 0$, where $x = tv + qv^\perp$ (we remind that ϑ_0 is defined by (3.24)). Then Proposition 4.3 follows from the definition (3.25). $\square$

Assume that (σ, k) satisfies condition (3.22) when $n = 2$ or (3.23) when $n \ge 2$. Assume also that $k(x,v,v') = k_0(x)g(v,v')$ for a.e. $(x,v,v') \in X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ where g is a given continuous function on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, $\inf_{v\in\mathbb{S}^{n-1}} g(v,v) > 0$, and where $k_0 \in L^\infty(X)$. Then from the decomposition of the angularly averaged albedo operator $A_{S,W}$ (Proposition 3.1) and from Theorems 3.2, 3.3, 3.4 and 3.5, and from Proposition 4.3 and methods of reconstruction of a function from its X-ray transform, it follows that (σ, k_0) can be reconstructed from the asymptotic expansion in time of $A_{S,W}$ provided that $\sigma = \sigma(x)$ and $\inf_{(x',v')\in\Gamma_-} S(x',v') > 0$ and $\inf_{(x,v)\in\Gamma_+} W(x,v) > 0$. In addition we have the following stability estimates.

Theorem 4.4. *Assume $X = B_n(0,1)$ and $\inf_{(x',v')\in\Gamma_-} S(x',v') > 0$ and $\inf_{(x,v)\in\Gamma_+} W(x,v) > 0$. Let (σ, k) and $(\tilde\sigma, \tilde k)$ satisfy either condition* (3.23) *or* (3.22)*. Assume that σ, $\tilde\sigma$ do not depend on the velocity variable ($\sigma(x,v) = \sigma(x)$) and* $\mathrm{supp}\sigma \cup \mathrm{supp}\tilde\sigma \subseteq Y$*, where $Y \subseteq X$ is a convex open subset of $\mathbb{R}^n$ with C^1 boundary, and let $M = \max(\|\sigma\|_{L^\infty(Y)}, \|\tilde\sigma\|_{L^\infty(Y)})$. Assume $k(x,v,v') = k_0(x)g(v,v')$ and $\tilde k(x,v,v') = \tilde k_0(x)g(v,v')$, $g(v,v) > 0$, for $(x,v,v') \in X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ where g is an a priori known continuous function on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$.*

Then there exists $C_3 = C_3(S, W, X, Y, M)$ such that

$$\|\sigma - \tilde{\sigma}\|_{H^{-\frac{1}{2}}(Y)} \le C_3 \|\sigma - \tilde{\sigma}\|_{L^\infty(Y)}^{\frac{1}{2}} \|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T}^{\frac{1}{2}}. \tag{4.11}$$

When $n \ge 2$ and (σ, k) satisfies (3.23), there exists $C_{4,1} = C_{4,1}(S, W, X, Y, Z, M, g)$ such that

$$\begin{aligned} \|\varrho(k_0 - \tilde{k}_0)\|_{H^{-\frac{1}{2}}(Z)} \le\ & C_{4,1} \|k_0 - \tilde{k}_0\|_\infty^{\frac{1}{2}} \Big(\|\tilde{k}_0\|_\infty \|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T} \\ & + \Big\| (\tau - |z - z'|)^{\frac{3-n}{2}} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \Big\|_{L^\infty} \Big)^{\frac{1}{2}}. \end{aligned} \tag{4.12}$$

When $n = 2$ and (σ, k) satisfies (3.22), there exists $C_{4,2} = C_{4,2}(S, W, X, M, g)$ such that

$$\begin{aligned} \|\varrho(k_0 - \tilde{k}_0)\|_{H^{-\frac{1}{2}}(X)} \le\ & C_{4,2} \|k_0 - \tilde{k}_0\|_\infty^{\frac{3}{4}} \Big(\|\tilde{k}_0\|_\infty \|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T} \\ & + \Big\| \sqrt{\tau^2 - |z - z'|^2} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \Big\|_{L^\infty} \Big)^{\frac{1}{4}}. \end{aligned} \tag{4.13}$$

Theorem 4.4 can be proved by mimicking the proof of Theorem 4.5 given below for a larger class of domains X. However we give a proof of estimate (4.13) in Section 6.

Note that the left-hand side $\|\varrho(k_0 - \tilde{k}_0)\|_{H^{-\frac{1}{2}}(Z)}$ of (4.12) can be replaced by $\|k_0 - \tilde{k}_0\|_{H^{-\frac{1}{2}}(Z)}$ since $\varrho^{-1} \in C^\infty(\bar{Z})$ and the operator $f \to \varrho^{-1} f$ is bounded in $H^{-\frac{1}{2}}(Z)$ for any open convex subset Z (with C^1 boundary) of X which satisfies $\bar{Z} \subseteq X$.

Under the assumptions of Theorem 4.4 and additional regularity assumptions on (σ, k) one obtains stability estimates similar to those given in Corollary 4.6 given below for a larger class of domains X.

4.3 Uniqueness and stability estimates for more general domains X

Theorem 4.5. *Assume that the open subset X of $\mathbb{R}^n$ is convex with a real analytic boundary and that $\inf_{(x',v') \in \Gamma_-} S(x', v') > 0$ and $\inf_{(x,v) \in \Gamma_+} W(x, v) > 0$. Let (σ, k) and $(\tilde{\sigma}, \tilde{k})$ satisfy condition (3.23). Assume also that σ, $\tilde{\sigma}$ do not depend on the velocity variable ($\sigma(x, v) = \sigma(x)$) and $k(x, v, v') = k_0(x) g(x, v, v')$ and $\tilde{k}(x, v, v') = \tilde{k}_0(x) g(x, v, v')$, $g(x, v, v') > 0$, for $(x, v, v') \in X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ where g is an a priori known real analytic function on $X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ and where $\mathrm{supp} k_0 \cup \mathrm{supp} \tilde{k}_0 \subseteq \bar{Z}$, $(k_0, \tilde{k}_0) \in L^\infty(Z)$. Then estimate (4.11) still holds and there exists $C_4 = C_4(S, W, X, Y, Z, M, g)$ such that*

$$\begin{aligned} \|k_0 - \tilde{k}_0\|_{H^{-\frac{1}{2}}(Z)} \le\ & C_4 \|k - \tilde{k}\|_\infty^{\frac{1}{2}} \Big(\|\tilde{k}\|_\infty \|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T} \\ & + \Big\| (\tau - |z - z'|)^{\frac{3-n}{2}} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \Big\|_{L^\infty} \Big)^{\frac{1}{2}}, \end{aligned} \tag{4.14}$$

where $M = \max(\|\sigma\|_{L^\infty(Y)}, \|\tilde{\sigma}\|_{L^\infty(Y)})$.

Theorem 4.5 is proved in Section 6. Assume that X is convex with a real analytic boundary and that $\inf_{(x',v') \in \Gamma_-} S(x', v') > 0$ and $\inf_{(x,v) \in \Gamma_+} W(x, v) > 0$. Let Y and Z be open convex subsets of X, $\bar{Z} \subset X$, $Z \subseteq Y \subseteq X$, with a C^1 boundary. Let g be an a priori known real analytic function on $X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, $g(x, v, v') > 0$ for $(x, v, v') \in X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. Let $r_1 > 0$, $r_2 > 0$. Consider the class

$$\begin{aligned} N := \ & \big\{ (\sigma, k) \in H^{\frac{n}{2} + r_1}(Y) \times L^\infty(Z \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}) \mid \|\sigma\|_{H^{\frac{n}{2} + r_1}(Y)} \le M_1, \\ & k = k_0 g,\ \mathrm{supp} k_0 \subseteq \bar{Z},\ \|k_0\|_{H^{\frac{n}{2} + r_2}(Z)} \le M_2 \big\}. \end{aligned} \tag{4.15}$$

Note that there exist a function $D_1 : \mathbb{N} \times (0,+\infty) \to (0,+\infty)$ such that

$$\begin{aligned} \|\sigma\|_{L^\infty(Y)} &\le D_1(n,r_1)\|\sigma\|_{H^{\frac{n}{2}+r_1}(Y)} \le D_1(n,r_1)M_1, \\ \|k_0\|_{L^\infty(Z)} &\le D_1(n,r_2)\|k_0\|_{H^{\frac{n}{2}+r_2}(Z)} \le D_1(n,r_2)M_2, \\ \|k\|_{L^\infty(Z)} &\le \|g\|_{L^\infty(Z)}\|k_0\|_{L^\infty(Z)} \le D_1(n,r_2)M_2\|g\|_{L^\infty(Z)}, \end{aligned} \tag{4.16}$$

for $(\sigma,k) \in N$. We also use the interpolation formula

$$\|f\|_{H^s(O)} \le \|f\|_{H^{s_1}(O)}^{\frac{s_2-s}{s_2-s_1}}\|f\|_{H^{s_2}(O)}^{\frac{s-s_1}{s_2-s_1}}, \tag{4.17}$$

for $s_1 < s < s_2$ and for $(O,s_1,s_2) \in \{(Y,-\frac{1}{2},\frac{n}{2}+r_1),\ (Z,-\frac{1}{2},\frac{n}{2}+r_2)\}$. Using Theorem 4.5 and (4.16), and applying (4.17) on $f = \sigma - \tilde{\sigma}$ and $f = k_0 - \tilde{k}_0$ we obtain the following result.

Corollary 4.6. *Let (σ,k), $(\tilde{\sigma},\tilde{k}) \in N$. Then, for $-\frac{1}{2} \le s \le \frac{n}{2}+r_1$ and for $0 < r < r_1$, there exists $C_5 = C_5(S,W,X,Y,M_1,r_1,s)$ such that*

$$\|\sigma-\tilde{\sigma}\|_{H^s(Y)} \le C_5\|\sigma-\tilde{\sigma}\|_{L^\infty(Y)}^{\frac{\kappa}{2}}\|A_{S,W}-\tilde{A}_{S,W}\|_{\eta,T}^{\frac{\kappa}{2}}, \tag{4.18}$$

$$\|\sigma-\tilde{\sigma}\|_{H^{\frac{n}{2}+r}(Y)} \le C_6\|A_{S,W}-\tilde{A}_{S,W}\|_{\eta,T}^{\frac{\kappa'}{2-\kappa'}}, \tag{4.19}$$

where $(\kappa,\kappa') = \left(\frac{n+2(r_1-s)}{n+1+2r_1}, \frac{2(r_1-r)}{n+1+2r_1}\right)$ and $C_6 = C_5^{\frac{2}{2-\kappa'}}D_1(n,r)^{\frac{\kappa'}{2-\kappa'}}$ ($D_1(n,r)$ is defined by (4.16)). In addition, there exists $C_7 = C_7(S,W,X,Y,Z,g,M_1,r_1,M_2,r_2,s)$ such that

$$\begin{aligned}\|k_0-\tilde{k}_0\|_{H^s(Z)} \le\ & C_7\|k_0-\tilde{k}_0\|_{L^\infty(Z)}^{\frac{\kappa}{2}}\Big(\|A_{S,W}-\tilde{A}_{S,W}\|_{\eta,T} \\ & +\left\|(\tau-|z-z'|)^{\frac{3-n}{2}}(\Gamma_1-\tilde{\Gamma}_1)(\tau,z,z')\right\|_{L^\infty}\Big)^{\frac{\kappa}{2}},\end{aligned} \tag{4.20}$$

$$\|k_0-\tilde{k}_0\|_{H^{\frac{n}{2}+r}(Z)} \le C_8\left(\|A_{S,W}-\tilde{A}_{S,W}\|_{\eta,T} + \left\|(\tau-|z-z'|)^{\frac{3-n}{2}}(\Gamma_1-\tilde{\Gamma}_1)(\tau,z,z')\right\|_{L^\infty}\right)^{\frac{\kappa'}{2-\kappa'}}, \tag{4.21}$$

for $-\frac{1}{2} \le s \le \frac{n}{2}+r_2$ and $0 < r < r_2$, where $(\kappa,\kappa') = \left(\frac{n+2(r_2-s)}{n+1+2r_2}, \frac{2(r_2-r)}{n+1+2r_2}\right)$ and $C_8 = C_7^{\frac{2}{2-\kappa'}}D_1(n,r)^{\frac{\kappa'}{2-\kappa'}}$ ($D_1(n,r)$ is defined by (4.16)).

Remark 4.7. (i.) Theorem 4.5 and Corollary 4.6 remain valid when: X is only assumed to be convex with C^2 boundary; the weight ϑ_o defined by (3.24) (resp. the function g which appears in the assumptions of Theorem 4.5 and Corollary 4.6) is sufficiently close (in the C^2 norm) to an analytic weight $\theta_{0,a}$ on the vicinity of $\bar{Z} \times \mathbb{S}^{n-1}$ (resp. an analytic function g_a on the vicinity of $\bar{Z} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$); see proof of Theorem 4.5 and [12, Theorem 2.3].

(ii.) When $n = 3$ then under hypothesis (3.23), we have

$$\left\|(\tau-|z-z'|)^{\frac{3-n}{2}}(\Gamma_1-\tilde{\Gamma}_1)(\tau,z,z')\right\|_{L^\infty} = \left\|\sum_{m=1}^{\infty}(A_{m,S,W}-\tilde{A}_{m,S,W})\right\|_{\mathcal{L}(L^1((0,\eta)\times\partial X),L^\infty((0,T)\times\partial X))}.$$

where the distributional kernel of the bounded operator $\sum_{m=1}^{+\infty}(A_{m,S,W}-\tilde{A}_{m,S,W})$ from $L^1((0,\eta)\times\partial X)$ to $L^1((0,T)\times\partial X)$ is given by $\Gamma_1-\tilde{\Gamma}_1$. Therefore when $n=3$ and under condition (3.23), the right-hand side of the stability estimates (4.14) and (4.20) can be expressed with operator norms only (instead of using a norm on the distributional kernel of the multiple scattering).

5 Proof of Theorems 3.4, 3.5

Proof of Theorem 3.5. For the sake of simplicity and without loss of generality we assume $v_0=(1,0,\ldots,0)$. Assume that condition (3.23) is satisfied. For $n\ge 2$ consider the following open subset of $(0,+\infty)\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$

$$\mathcal{D}:=\{(s,v,v')\in(0,+\infty)\times\mathbb{S}^{n-1}_{x,+}\times\mathbb{S}^{n-1}_{x_0',-}\mid s\in(0,\tau_-(x,v))\}. \tag{5.1}$$

Then we introduce the bounded function Ψ_n on $\mathcal{D}$ defined by

$$\Psi_n(s,v,v')=2^{n-2}W(x,v)(\nu(x)\cdot v)E(x,x-sv,x_0')k(x-sv,v',v)S(x_0',v')|\nu(x_0')\cdot v'|, \tag{5.2}$$

for $(s,v,v')\in\mathcal{D}$. Note that from convexity of X it follows that $\tau_\pm$ is continuous on $\Gamma_\mp$ and $E(x,x-sv,x_0')=e^{-\int_0^s\sigma(x-pv,v)dp-\int_0^{|x-x_0'-sv|}\sigma(x-sv-p\frac{x-x_0'-sv}{|x-x_0'-sv|},\frac{x-x_0'-sv}{|x-x_0'-sv|})dp}$ for $v\in\mathbb{S}^{n-1}_{x,+}$ and $0<s<\tau_-(x,v)$. Under (3.23) we obtain that

$$\begin{aligned}&\Psi_n(s,v,v')=0\text{ for }(s,v,v')\in(0,+\infty)\times\mathbb{S}^{n-1}_{x,+}\times\mathbb{S}^{n-1}_{x_0',-}\text{ such that }x-sv\notin\bar Z,\\ &\text{and the function }\Psi_n\text{ is continuous at any point }(s,v,v')\in\mathcal{D}\text{ such that }x-sv\in Z.\end{aligned} \tag{5.3}$$

We first prove (3.26) for $n=2$. Let $\tau>t_0$. From (5.2), (3.10), it follows that

$$\begin{aligned}\gamma_1(\tau,x,x_0') &= \int_{-\alpha_0}^{\pi-\alpha_0}\frac{1}{\tau-t_0\cos(\Omega)}\left[\chi_{(0,\tau_-(x,v))}(s)\Psi_2(s,v,v')\right]_{\substack{v=(\cos\Omega,\sin\Omega)\\ v'=\frac{t_0(1,0)-sv}{\tau-s}\\ s=\frac{\tau^2-t_0^2}{2(\tau-t_0\cos(\Omega))}}}d\Omega\\ &= \gamma_{1,1}(\tau,x,x_0')+\gamma_{1,2}(\tau,x,x_0'),\end{aligned} \tag{5.4}$$

where $\mathbb{S}^{n-1}_{x,+}=\{(\cos\Omega,\sin\Omega)\mid -\alpha_0<\Omega<\pi-\alpha_0\}$ $(0<\alpha_0<\pi)$ and

$$\gamma_{1,1}(\tau,x,x_0') = \int_0^{\pi}\frac{\chi_{(0,\pi-\alpha_0)}(\Omega)}{\tau-t_0\cos(\Omega)}\left[\chi_{(0,\tau_-(x,v))}(s)\Psi_2(s,v,v')\right]_{\substack{v=(\cos\Omega,\sin\Omega)\\ v'=\frac{t_0(1,0)-sv}{\tau-s}\\ s=\frac{\tau^2-t_0^2}{2(\tau-t_0\cos(\Omega))}}}d\Omega, \tag{5.5}$$

$$\gamma_{1,2}(\tau,x,x_0') = \int_{-\pi}^{0}\frac{\chi_{(-\alpha_0,0)}(\Omega)}{\tau-t_0\cos(\Omega)}\left[\chi_{(0,\tau_-(x,v))}(s)\Psi_2(s,v,v')\right]_{\substack{v=(\cos\Omega,\sin\Omega)\\ v'=\frac{t_0(1,0)-sv}{\tau-s}\\ s=\frac{\tau^2-t_0^2}{2(\tau-t_0\cos(\Omega))}}}d\Omega. \tag{5.6}$$

We shall prove that

$$\begin{aligned}&\sqrt{\tau-t_0}\gamma_{1,i}(\tau,x,x_0')\to\frac{W(x,v_0)S(x_0',v_0)(\nu(x)\cdot v_0)|\nu(x_0')\cdot v_0|E(x_0',x)}{\sqrt{2t_0}}\int_0^{t_0}\frac{k(x-sv_0,v_0,v_0)}{\sqrt{s(t_0-s)}}ds,\\ &\text{as }\tau\to t_0^+,\end{aligned} \tag{5.7}$$

for $i=1,2$. Then adding (5.7) for $i=1$ and $i=2$, we obtain (3.26). We only prove (5.7) for $i=1$ since the proof for $i=2$ is similar. Let $\tau>t_0$. Using the change of variables $s=\frac{\tau^2-t_0^2}{2(\tau-t_0\cos(\Omega))}-\frac{\tau-t_0}{2}$, we obtain

$$\gamma_{1,1}(\tau,x,x_0')=\frac{1}{\sqrt{\tau^2-t_0^2}}\int_0^{t_0}\chi_{(0,\pi-\alpha_0)}(\Omega(s,\tau))\frac{\chi_{(0,\tau_-(x,v(s,\tau)))}(s+\frac{\tau-t_0}{2})\Psi_2(s,v(s,\tau),v'(s,\tau))}{\sqrt{s(t_0-s)}}d\tau, \tag{5.8}$$

where

$$
\begin{aligned}
&v(s,\tau) = (\cos\Omega(s,\tau), \sin\Omega(s,\tau)),\ \ \Omega(s,\tau) = \arccos\left(\frac{\tau - \frac{\tau^2 - t_0^2}{2s+\tau-t_0}}{t_0}\right),\\
&v'(s,\tau) = \frac{t_0(1,0) - \left(s + \frac{\tau - t_0}{2}\right) v(s,\tau)}{\frac{\tau+t_0}{2} - s}.
\end{aligned}
\tag{5.9}
$$

Let $s \in (0, t_0)$. From (5.9), it follows that

$$
v(s,\tau) \to (1,0) \text{ as } \tau \to t_0^+,\ \ v'(s,\tau) \to (1,0) \text{ as } \tau \to t_0^+. \tag{5.10}
$$

Note that using the definition of v_0 and using the assumption $x_0' + \varepsilon(x - x_0') \in X$ for some $\varepsilon \in (0,1)$ we obtain $t_0 = \tau_-(x, v_0)$. Note also that the function $s \mapsto \frac{1}{\sqrt{s(t_0-s)}}$, $s \in (0, t_0)$, is integrable in $(0, t_0)$. Therefore, using (5.3), the boundedness of Ψ_2 on $\mathcal{D}$ and the Lebesgue dominated convergence theorem, we obtain (5.7). This proves (3.26) when $n = 2$.

Let $n \geq 3$ and prove (3.29). From (5.2) and (3.10), it follows that

$$
\gamma_1(\tau, x, x_0') = \int_{\mathbb{S}^{n-1}} \frac{(\tau - t_0 v_0 \cdot v)^{n-3}}{|t_0 v_0 - \tau v|^{2n-4}} \chi_{(0,+\infty)}(\nu(x)\cdot v) \Psi_n(s, v, v')\Big|_{\substack{v' = \frac{t_0 v_0 - s v}{\tau - s} \\ s = \frac{\tau^2 - t_0^2}{2(\tau - t_0 v \cdot v_0)}}} dv, \tag{5.11}
$$

for $\tau > |x - x_0'|$.

Let $\Phi(\Omega, \omega) = (\sin\Omega, \cos(\Omega)\omega_1, \ldots, \cos(\Omega)\omega_{n-1})$ for $\Omega \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\omega = (\omega_1, \ldots, \omega_{n-1}) \in \mathbb{S}^{n-2}$. Using spherical coordinates we obtain

$$
\begin{aligned}
\gamma_1(\tau, x, x_0') \ = \ & \int_{-\pi/2}^{\pi/2} \cos(\Omega)^{n-2} \frac{(\tau - t_0 \sin(\Omega))^{n-3}}{(t_0^2 + \tau^2 - 2t_0\tau\sin(\Omega))^{n-2}} \\
& \int_{\mathbb{S}^{n-2}} \chi_{(0,+\infty)}(\nu(x)\cdot\Phi(\Omega,\omega)) \Psi_n(s, \Phi(\Omega,\omega), v')\Big|_{\substack{v' = \frac{t_0 v_0 - s\Phi(\Omega,\omega)}{\tau - s} \\ s = \frac{\tau^2 - t_0^2}{2(\tau - t_0 \sin(\Omega))}}} d\omega d\Omega,
\end{aligned}
\tag{5.12}
$$

for $\tau > t_0$. Performing the change of variables "$r = \frac{\tau^2 - t_0^2}{2(\tau - t_0\sin(\Omega))} - \frac{\tau - t_0}{2}$" on the first integral on the right-hand side of (5.12), we obtain

$$
\begin{aligned}
\gamma_1(\tau, x, x_0') \ = \ & 2^{2-n} t_0^{2-n} (\tau^2 - t_0^2)^{\frac{n-3}{2}} \int_0^{t_0} \frac{\sqrt{r(t_0 - r)}^{n-3}}{(\frac{\tau - t_0}{2} + r)^{n-2} (\frac{\tau + t_0}{2} - r)^{n-2}} \\
& \int_{\mathbb{S}^{n-2}} \left[\chi(\Phi(\Omega,\omega)) \Psi_n(r + \frac{\tau - t_0}{2}, \Phi(\Omega,\omega), v')\right]_{\substack{\Omega = \arcsin(t_0^{-1}(\tau - \frac{(\tau^2 - t_0^2)}{2(r + \frac{\tau - t_0}{2})})) \\ s = r + \frac{\tau - t_0}{2} \\ v' = \frac{t_0 v_0 - s\Phi(\Omega,\omega)}{\tau - s}}} d\omega dr.
\end{aligned}
\tag{5.13}
$$

Therefore using (5.13), (5.3) and (5.2) and using Lebesgue dominated convergence theorem, we obtain (3.29). This concludes the proof of Theorem 3.5. $\square$

Proof of Theorem 3.4. For the sake of simplicity and without loss of generality we assume $v_0 = (1, 0, \ldots, 0)$. Assume that condition (3.22) is satisfied. We consider the measurable function Ψ_n defined by (5.2) for all $(s, v, v') \in \mathcal{D}$ where $\mathcal{D}$ is defined by (5.1). Under (3.22) we obtain that

$$
\begin{aligned}
&\text{the function } \Psi_n \text{ is continuous at any point } (s, v, v') \in \mathcal{D} \\
&\text{(i.e. for any } (s, v, v') \in (0, +\infty) \times \mathbb{S}_x^{n-1} \times \mathbb{S}_{x_0'}^{n-1},\ x - sv \in X),
\end{aligned}
\tag{5.14}
$$

The proof of (3.26) under condition (3.22) is actually similar to the proof of (3.26) under condition (3.23). Note that (5.4)–(5.6) still hold so that we have to prove that (5.7) still holds for $i=1,2$. Again we only sketch the proof of (5.7) for $i=1$. Note also that (5.8)–(5.10) still hold. Then using (5.8)–(5.10), (5.14) and (5.2) and using Lebesgue dominated convergence theorem, we obtain (5.7) for $i=1$. This proves (3.26).

Let $n\geq 3$. Formula (5.12) still holds. Now assume that $n=3$. We shall prove (3.27). Let $\tau>t_0$. Using the change of variables "$\varepsilon=\frac{\ln(t_0^2+\tau^2-2t_0\tau\sin(\Omega))-\ln((\tau-t_0)^2)}{\ln((\tau+t_0)^2)-\ln((\tau-t_0)^2)}$", the equality (5.12) gives

$$\gamma_1(\tau,x,x_0')=\frac{\ln\left(\frac{\tau+t_0}{\tau-t_0}\right)}{2t_0\tau}(\gamma_{1,1}(\tau,x,x_0')+\gamma_{1,2}(\tau,x,x_0')) \tag{5.15}$$

where

$$\gamma_{1,1}(\tau,x,x_0')=\int_0^{\frac{1}{2}}\int_{\mathbb{S}^1}\chi_{(0,+\infty)}(\nu(x)\cdot\Phi(\Omega(\tau,\varepsilon),\omega))\Psi_3(s(\tau,\varepsilon),\Phi(\Omega(\tau,\varepsilon),\omega),v'(\tau,\omega,\varepsilon))d\omega d\varepsilon \tag{5.16}$$

$$\gamma_{1,2}(\tau,x,x_0')=\int_{\frac{1}{2}}^{1}\int_{\mathbb{S}^1}\chi_{(0,+\infty)}(\nu(x)\cdot\Phi(\Omega(\tau,\varepsilon),\omega))\Psi_3(s(\tau,\varepsilon),\Phi(\Omega(\tau,\varepsilon),\omega),v'(\tau,\omega,\varepsilon))d\omega d\varepsilon \tag{5.17}$$

and

$$\Omega(\tau,\varepsilon) := \arcsin\left(\frac{\tau^2+t_0^2-(\tau-t_0)^{2(1-\varepsilon)}(\tau+t_0)^{2\varepsilon}}{2t_0\tau}\right), \tag{5.18}$$

$$s(\tau,\varepsilon) := \frac{\tau^2-t_0^2}{2(\tau-t_0\sin(\Omega(\tau,\varepsilon)))}, \tag{5.19}$$

$$v'(\tau,\omega,\varepsilon) := \frac{t_0v_0-s(\tau,\varepsilon)\Phi(\Omega(\tau,\varepsilon),\omega)}{\tau-s(\tau,\varepsilon)}, \tag{5.20}$$

for $0<\varepsilon<1$, $\omega\in\mathbb{S}^1$.

We shall give some properties of $\Omega(\tau,\varepsilon)$, $s(\tau,\varepsilon)$ and $v'(\tau,\omega,\varepsilon)$ for $0<\varepsilon<1$ and $\omega\in\mathbb{S}^{n-1}$. From (5.18), it follows that

$$\Omega(\tau,\varepsilon)\to\frac{\pi}{2},\ \text{as}\ \tau\to t_0^+,\ \text{for all}\ \varepsilon\in(0,1). \tag{5.21}$$

From (5.18) and (5.19), it follows that

$$s(\tau,\varepsilon)=\frac{\tau(t_0+\tau)(\tau-t_0)^{2\varepsilon-1}}{(t_0+\tau)(\tau-t_0)^{2\varepsilon-1}+(\tau+t_0)^{2\varepsilon}},\ s(\tau,\varepsilon)=\frac{\tau(t_0+\tau)}{t_0+\tau+(\tau-t_0)^{1-2\varepsilon}(\tau+t_0)^{2\varepsilon}} \tag{5.22}$$

for $\varepsilon\in(0,1)$. From (5.22) it follows that

$$s(\tau,\varepsilon)\to 0^+\ \text{as}\ \tau\to t_0^+,\ \text{when}\ \varepsilon\in(1/2,1),\ s(\tau,\varepsilon)\to t_0^-\ \text{as}\ \tau\to t_0^+,\ \text{when}\ \varepsilon\in(0,1/2). \tag{5.23}$$

In addition, from (5.18)–(5.20), it follows that

$$\begin{aligned} v'(\tau,\omega,\varepsilon) \;=\; & \left(\frac{-\tau(\tau+t_0)^2(\tau-t_0)^{2\varepsilon}+2t_0^2\tau(\tau+t_0)^{2\varepsilon}+\tau(\tau+t_0)^{1+2\varepsilon}(\tau-t_0)}{2t_0\tau^2(\tau+t_0)^{2\varepsilon}},\right.\\ & \sqrt{-(\tau-t_0)^{4\varepsilon}(\tau+t_0)^2-(\tau-t_0)^2(\tau+t_0)^{4\varepsilon}+2(\tau^2+t_0^2)(\tau-t_0)^{2\varepsilon}(\tau+t_0)^{2\varepsilon}}\\ & \left.\times\frac{(\tau+t_0+(\tau-t_0)^{1-2\varepsilon}(\tau+t_0)^{2\varepsilon})}{2t_0\tau(\tau+t_0)^{2\varepsilon}}\omega\right) \end{aligned} \tag{5.24}$$

for $0<\varepsilon<1$. Therefore

$$v'(\tau,\omega,\varepsilon)\to v_0=(1,0,0) \text{ as } \tau\to t_0^+, \tag{5.25}$$

for $0<\varepsilon<1$.

Using (5.16), (5.21), (5.23), (5.25) and Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}\gamma_{1,1}(\tau,x,x_0') &\underset{\tau\to t_0^+}{\to} \int_0^{\frac12}\int_{\mathbb{S}^1}\lim_{s\to t_0^-}\Psi_3(s,v_0,v_0)d\omega d\varepsilon\\ &= 2\pi E(x_0',x)W(x,v_0)S(x_0',v_0)(\nu(x_0')v_0)(\nu(x)v_0)k(x_0',v_0,v_0) \end{aligned}\tag{5.26}$$

(we also used (5.2)).

Similarly, using (5.16), (5.21), (5.23), (5.25) and Lebesgue dominated convergence theorem, we obtain

$$\gamma_{1,2}(\tau,x,x_0')\to 2\pi E(x_0',x)W(x,v_0)S(x_0',v_0)(\nu(x_0')v_0)(\nu(x)v_0)k(x,v_0,v_0),\ \text{ as } \tau\to t_0^+. \tag{5.27}$$

Statement (3.27) follows from (5.15) and (5.26)–(5.27).

Let $n\ge 4$. We shall prove (3.28). From (5.12) it follows that

$$\gamma_1(\tau,x,x_0')=\gamma_{1,1}(\tau,x,x_0')+\gamma_{1,2}(\tau,x,x_0'), \tag{5.28}$$

where

$$\begin{aligned}\gamma_{1,1}(\tau,x,x_0') &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\chi_{\left(-\frac{\pi}{2},\arcsin\left(\frac{t_0}{\tau}\right)\right)}(\Omega)\cos(\Omega)^{n-2}\frac{(\tau-t_0\sin(\Omega))^{n-3}}{(t_0^2+\tau^2-2t_0\tau\sin(\Omega))^{n-2}} \\ &\times\int_{\mathbb{S}^{n-2}}\chi_{(0,+\infty)}(\Phi(\Omega,\omega)\cdot\nu(x))\chi_{(0,\tau_-(x,\Phi(\Omega,\omega))))}(s)\Psi_n(s,\Phi(\Omega,\omega),v')_{\substack{v'=\frac{t_0v_0-s\Phi(\Omega,\omega)}{\tau-s}\\ s=\frac{\tau^2-t_0^2}{2(\tau-t_0\sin(\Omega))}}}d\omega d\Omega,\end{aligned}\tag{5.29}$$

$$\begin{aligned}\gamma_{1,2}(\tau,x,x_0') &= \int_{\arcsin\left(\frac{t_0}{\tau}\right)}^{\frac{\pi}{2}}\cos(\Omega)^{n-2}\frac{(\tau-t_0\sin(\Omega))^{n-3}}{(t_0^2+\tau^2-2t_0\tau\sin(\Omega))^{n-2}} \\ &\times\int_{\mathbb{S}^{n-2}}\chi_{(0,+\infty)}(\nu(x)\cdot\Phi(\Omega,\omega))\chi_{(0,\tau_-(x,\Phi(\Omega,\omega))))}(s)\Psi_n(s,\Phi(\Omega,\omega),v')_{\substack{v'=\frac{t_0v_0-s\Phi(\Omega,\omega)}{\tau-s}\\ s=\frac{\tau^2-t_0^2}{2(\tau-t_0\sin(\Omega))}}}d\omega d\Omega.\end{aligned}\tag{5.30}$$

First we study $\gamma_{1,1}$. Note that

$$\frac{\tau-t_0\sin(\Omega_1)}{t_0^2+\tau^2-2t_0\tau\sin(\Omega_1)}<\frac{\tau-t_0\sin(\Omega_2)}{t_0^2+\tau^2-2t_0\tau\sin(\Omega_2)}, \tag{5.31}$$

for $-\frac{\pi}{2}\le\Omega_1<\Omega_2\le\frac{\pi}{2}$ and for $\tau>t_0$. Therefore using also the estimate $\cos(\Omega)\le 1$ we obtain

$$\begin{aligned}&\cos(\Omega)^{n-2}\frac{(\tau-t_0\sin(\Omega))^{n-3}}{(t_0^2+\tau^2-2t_0\tau\sin(\Omega))^{n-2}}\\ &\le\cos(\Omega)^{n-4}\frac{\cos(\Omega)^2}{t_0^2+\tau^2-2t_0\tau\sin(\Omega)}\left(\frac{\tau-t_0\sin(\Omega)}{t_0^2+\tau^2-2t_0\tau\sin(\Omega)}\right)^{n-3}\le\frac{C_0^2}{2t_0\tau^{n-2}},\end{aligned}\tag{5.32}$$

for $\Omega\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $\sin(\Omega)\le\frac{t_0}{\tau}$ (we used (5.31) with "Ω_1"$=\Omega$ and "Ω_2"$=\frac{t_0}{\tau}$, and we used the estimate $t_0^2+\tau^2-2t_0\tau\sin(\Omega)\ge 2t_0\tau(1-\sin(\Omega))$), where

$$C_0:=\sup_{\varphi\in(0,2\pi)}\frac{\sin^2(\varphi)}{1-\cos(\varphi)}=\sup_{\Omega\in(-\frac{3\pi}{2},\frac{\pi}{2})}\frac{\cos^2(\Omega)}{1-\sin(\Omega)}. \tag{5.33}$$

Using (5.29), (5.32), (5.2) and Lebesgue dominated convergence theorem, we obtain

$$\gamma_{1,1}(\tau,x,x_0') \underset{\tau\to t_0^+}{\longrightarrow} 2^{2-n}t_0^{1-n}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos(\Omega)^{n-2}}{1-\sin(\Omega)}\int_{\mathbb{S}^{n-2}}\chi(\nu(x)\cdot\Phi(\Omega,\omega))\lim_{s\to 0^+}\Psi_n(s,\Phi(\Omega,\omega),v_0)d\omega d\Omega$$
$$= 2^{2-n}t_0^{1-n}\int_{\mathbb{S}^{n-1}_{x,+}}\frac{1}{1-v\cdot v_0}\lim_{s\to 0^+}\Psi_n(s,v,v_0)dv$$
$$= t_0^{1-n}E(x_0',x)S(x_0',v_0)(\nu(x_0')\cdot v_0)\int_{\mathbb{S}^{n-1}_{x,+}}\frac{1}{1-v\cdot v_0}W(x,v)(\nu(x)\cdot v)k(x,v_0,v)dv. \tag{5.34}$$

Now we shall study $\gamma_{1,2}$ defined by (5.30). Note that using the convexity of X we obtain $v'=\frac{x-sv-x_0'}{|x-sv-x_0'|}\in\mathbb{S}^{n-1}_{x_0',-}$ whenever $v\in\mathbb{S}^{n-1}_{x,+}$ and $s\in(0,\tau_-(x,v))$. Therefore using the change of variables "$\sin(\Omega')=\frac{2\tau t_0-(\tau^2+t_0^2)\sin(\Omega)}{\tau^2+t_0^2-2t_0\tau\sin(\Omega)}$", $\Omega\in(-\frac{\pi}{2},\frac{\pi}{2})$ ("$\cos(\Omega)\frac{d\Omega}{d\Omega'}=\frac{(\tau^2-t_0^2)^2\cos(\Omega')}{(\tau^2+t_0^2-2t_0\tau\sin(\Omega'))^2}$"), we obtain

$$\gamma_{1,2}(\tau,x,x_0') := \int_{-\frac{\pi}{2}}^{\arcsin\left(\frac{t_0}{\tau}\right)}\cos(\Omega')^{n-2}\frac{(\tau-t_0\sin(\Omega'))^{n-3}}{(t_0^2+\tau^2-2t_0\tau\sin(\Omega'))^{n-2}} \tag{5.35}$$
$$\int_{\substack{\omega\in\mathbb{S}^{n-2}\\ \nu(x)\cdot\Phi(\Omega(\tau,\Omega'),\omega)>0\\ -\nu(x_0')\cdot\Phi(\Omega',-\omega)>0}}\chi_{(0,\tau_-(x,\Phi(\Omega(\tau,\Omega'),\omega)))}(s(\tau,\Omega'))\Psi_n(s(\tau,\Omega'),\Phi(\Omega(\tau,\Omega'),\omega),v')_{v'=\Phi(\Omega',-\omega)}d\omega d\Omega',$$

where

$$\Omega(\tau,\Omega')=\arcsin\left(\frac{2\tau t_0-(\tau^2+t_0^2)\sin(\Omega')}{\tau^2+t_0^2-2t_0\tau\sin(\Omega')}\right),\ s(\tau,\Omega')=\frac{\tau^2+t_0^2-2t_0\tau\sin(\Omega')}{2(\tau-t_0\sin(\Omega'))}, \tag{5.36}$$

for $\Omega'\in(-\frac{\pi}{2},\frac{\pi}{2})$. Note that

$$\Omega(\tau,\Omega') \underset{\tau\to t_0^+}{\to} \frac{\pi}{2},\ \Phi(\Omega(\tau,\Omega'),\omega) \underset{\tau\to t_0^+}{\to} v_0, \tag{5.37}$$
$$s(\tau,\Omega') \underset{\tau\to t_0^+}{\to} t_0, \tag{5.38}$$

for $(\Omega',\omega)\in(-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{S}^{n-2}$. Note also that from (5.37), it follows that at fixed $v'=\Phi(\Omega',-\omega)\in\mathbb{S}^{n-1}_{x_0',-}$ the condition $\chi_{(0,\tau_-(x,\Phi(\Omega(\tau,\Omega'),\omega)))}(s(\tau,\Omega'))=1$ (which is equivalent to $x-s(\tau,\Omega')\Phi(\Omega(\tau,\Omega'),\omega)\in X$ due to convexity of X) for $\Phi(\Omega(\tau,\Omega'),\omega)\in\mathbb{S}^{n-1}_{x,+}$ is satisfied when $\tau-t_0>0$ is sufficiently small. Therefore from (5.32) (with "Ω" replaced by "Ω'") and from (5.35) and Lebesgue dominated convergence theorem we obtain

$$\gamma_{1,2}(\tau,x,x_0') \underset{\tau\to t_0^+}{\longrightarrow} 2^{2-n}t_0^{1-n}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos(\Omega')^{n-2}}{1-\sin(\Omega')}\int_{\mathbb{S}^{n-2}}\chi_{(0,+\infty)}(-\nu(x_0')\cdot\Phi(\Omega',-\omega))$$
$$\times\lim_{s\to t_0^-}\Psi_n(s,v_0,\Phi(\Omega',-\omega))d\omega d\Omega' = 2^{2-n}t_0^{1-n}\int_{\mathbb{S}^{n-1}_{x_0',-}}\frac{1}{1-v'\cdot v_0}\lim_{s\to t_0^-}\Psi_n(s,v_0,v')dv'$$
$$= t_0^{1-n}E(x_0',x)W(x,v_0)(\nu(x)\cdot v_0)\int_{\mathbb{S}^{n-1}_{x_0',-}}\frac{1}{1-v'\cdot v_0}S(x_0',v')|\nu(x_0')\cdot v'|k(x,v',v_0)dv'. \tag{5.39}$$

Statement (3.28) follows from (5.28), (5.34) and (5.39). Theorem 3.4 is proved. □

6 Proof of Theorems 4.1, 4.2, 4.5 and Theorem 4.4 (4.13)

Proof of Theorem 4.2. We now prove (4.5). Let $x'_0 \in \partial X$. For $\varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty)^2$ and $\varepsilon_3 \in (0, +\infty)$ let $(f_{\varepsilon_1}, g_{\varepsilon_2}) \in C^1(\partial X) \times C^1(\mathbb{R})$ satisfy

$$g_{\varepsilon_2} \geq 0,\, f_{\varepsilon_1} \geq 0,\ \mathrm{supp} g_{\varepsilon_2} \subseteq (0, \min(\varepsilon_2, \eta)), \tag{6.1}$$

$$\mathrm{supp} f_{\varepsilon_1} \subseteq \{x' \in \partial X \mid |x' - x'_0| < \varepsilon_1\}, \tag{6.2}$$

$$\int_0^\eta g_{\varepsilon_2}(t')dt' = 1,\ \int_{\partial X} f_{\varepsilon_1}(x')d\mu(x') = 1, \tag{6.3}$$

for $\varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty)^2$. Therefore $\phi_\varepsilon := g_{\varepsilon_2} f_{\varepsilon_1}$ is an approximation of the delta function at $(0, x'_0) \in \mathbb{R} \times \partial X$ for $\varepsilon := (\varepsilon_1, \varepsilon_2) \in (0, +\infty)^2$. Let $\psi_{\varepsilon_3} \in L^\infty((0,T) \times \partial X)$ be defined by

$$\psi_{\varepsilon_3}(t,x) = \chi_{(-\varepsilon_3, \varepsilon_3)}(t - |x - x'_0|)(2\chi_{(0,+\infty)}((E - \tilde{E})(x, x'_0)) - 1),\ (t,x) \in (0,T) \times \partial X, \tag{6.4}$$

for $\varepsilon_3 > 0$. From (3.8) and (3.12) it follows that

$$\begin{aligned}\int_{(0,T)\times\partial X} \psi_{\varepsilon_3}(t,x)(A_{S,W} - \tilde{A}_{S,W})\phi_\varepsilon(t,x)dtd\mu(x) = I_0(\psi_{\varepsilon_3}, \phi_\varepsilon)\\ + \int_{(0,T)\times\partial X\times(0,\eta)\times\partial X} \psi_{\varepsilon_3}(t,x)\phi_\varepsilon(t',x')(\Gamma_1 - \tilde{\Gamma}_1)(t - t', x, x')dtd\mu(x)dt'd\mu(x'),\end{aligned} \tag{6.5}$$

for $\varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty)$ and $\varepsilon_3 \in (0, +\infty)$, where

$$\begin{aligned} I_0(\psi_{\varepsilon_3}, \phi_\varepsilon) &= \int_{\substack{(0,T)_t\times\partial X_x\times\partial X_{x'}\\ |x-x'|<t}} \psi_{\varepsilon_3}(t,x)\phi_\varepsilon(t - |x-x'|, x')\frac{E(x,x') - \tilde{E}(x,x')}{|x-x'|^{n-1}}\\ &\times [W(x,v)S(x',v)(\nu(x)\cdot v)|\nu(x')\cdot v|]_{v = \frac{x-x'}{|x-x'|}}\, dtd\mu(x)d\mu(x').\end{aligned} \tag{6.6}$$

From (3.16), (3.17), (3.20) and (3.21) it follows that

$$(\tau - |x - x'|)^{\frac{3-n}{2}}(\Gamma_1 - \tilde{\Gamma}_1)(\tau, x, x') \in L^\infty((0,T) \times \partial X \times \partial X). \tag{6.7}$$

Combining (6.5) and the equality $\|\phi_\varepsilon\|_{L^1((0,\eta)\times\partial X)} = 1$ and the estimate $\|\psi_{\varepsilon_3}\|_{L^\infty((0,T)\times\partial X)} \leq 1$ and (6.7) we obtain

$$I_0(\psi_{\varepsilon_3}, \phi_\varepsilon) \leq \|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T} + C\Delta_1(\psi_{\varepsilon_3}, \phi_\varepsilon), \tag{6.8}$$

for $\varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty)$ and $\varepsilon_3 \in (0, +\infty)$, where $C = \|(\tau - |x - x'|)^{\frac{3-n}{2}}(\Gamma_1 - \tilde{\Gamma}_1)(\tau, x, x')\|_{L^\infty((0,T)\times\partial X_x\times\partial X_{x'})}$ and

$$\Delta_1(\psi_{\varepsilon_3}, \phi_\varepsilon) = \int_{\substack{(0,T)_t\times\partial X_x\times(0,\eta)_{t'}\times\partial X_{x'}\\ |x-x'|<t-t'}} \psi_{\varepsilon_3}(t,x)\phi_\varepsilon(t',x')(t - t' - |x-x'|)^{\frac{n-3}{2}}dtd\mu(x)dt'd\mu(x'). \tag{6.9}$$

Note that the function $\Phi_{1,\varepsilon_3} : [0,\eta) \times \partial X \to \mathbb{R}$ defined by

$$\Phi_{1,\varepsilon_3}(t', x') := \int_{\substack{(0,T)_t\times\partial X_x\\ |x-x'|<t-t'}} \psi_{\varepsilon_3}(t,x)(t - t' - |x-x'|)^{\frac{n-3}{2}}dtd\mu(x),\ (t',x') \in [0,\eta) \times \partial X, \tag{6.10}$$

is continuous on $[0,\eta) \times \partial X$ for $\varepsilon_3 \in (0, +\infty)$. Therefore from (6.1)–(6.3) and the equality $\Delta_1(\psi_{\varepsilon_3}, \phi_\varepsilon) = \int_{(0,\eta)\times\partial X} \phi_\varepsilon(t',x')\Phi_{1,\varepsilon_3}(t',x')dt'd\mu(x')$ it follows that

$$\lim_{\varepsilon_3\to0^+}\lim_{\varepsilon_2\to0^+}\lim_{\varepsilon_1\to0^+}\Delta_1(\psi_{\varepsilon_3}, \phi_\varepsilon) = \lim_{\varepsilon_3\to0^+}\Phi_{1,\varepsilon_3}(0, x'_0) = 0 \tag{6.11}$$

(we also used (6.10), (6.4) and the Lebesgue dominated convergence theorem to prove that $\lim_{\varepsilon_3\to 0^+}\Phi_{1,\varepsilon_3}(0,x_0')=0$). Note that under condition (3.23) the function $\Phi_{0,\varepsilon_2,\varepsilon_3}:\partial X\to\mathbb{R}$ defined by

$$\begin{aligned}\Phi_{0,\varepsilon_2,\varepsilon_3}(x') &= \int_{\substack{(0,T)_t\times\partial X_x\\ |x-x'|<t}}\psi_{\varepsilon_3}(t,x)g_{\varepsilon_2}(t-|x-x'|)\frac{E(x,x')-\tilde E(x,x')}{|x-x'|^{n-1}}\\ &\quad\times\left[W(x,v)S(x',v)(\nu(x)\cdot v)|\nu(x')\cdot v|\right]_{v=\frac{x-x'}{|x-x'|}}dtd\mu(x),\end{aligned}\tag{6.12}$$

is continuous on ∂X, for $(\varepsilon_2,\varepsilon_3)\in(0,+\infty)^2$. Therefore from the equality $I_0(\psi_{\varepsilon_3},\phi_\varepsilon)=\int_{\partial X}\Phi_{0,\varepsilon_2,\varepsilon_3}(x')\times f_{\varepsilon_1}(x')d\mu(x')$ (see (6.6)) it follows that

$$\lim_{\varepsilon_1\to 0^+}I_0(\psi_{\varepsilon_3},\phi_\varepsilon)=\Phi_{0,\varepsilon_2,\varepsilon_3}(x_0'),\ \text{for } (\varepsilon_2,\varepsilon_3)\in(0,+\infty)^2.\tag{6.13}$$

Therefore using the Lebesgue dominated convergence theorem and (6.12) we obtain

$$\lim_{\varepsilon_3\to 0^+}\lim_{\varepsilon_2\to 0^+}\lim_{\varepsilon_1\to 0^+}I_0(\psi_{\varepsilon_3},\phi_\varepsilon)=\int_{\partial X_x}\frac{\left|E(x,x_0')-\tilde E(x,x_0')\right|}{|x-x_0'|^{n-1}}\left[W(x,v)S(x_0',v)(\nu(x)\cdot v)|\nu(x_0')\cdot v|\right]_{v=\frac{x-x_0'}{|x-x_0'|}}d\mu(x).\tag{6.14}$$

Combining (6.14), (6.11) and (6.8) we obtain the formula (4.1). Using (4.1) and the estimates $\inf_{(x',v')\in\Gamma_-}S(x',v')>0$ and $\inf_{(x,v)\in\Gamma_+}W(x,v)>0$ and the change of variables $x=x_0'+\tau_+(x_0',v_0)v_0$ ($\frac{\nu(x)\cdot v_0}{|x-x_0'|^{n-1}}d\mu(x)=dv_0$) we obtain (4.5) where the constant C_1 which appears on the right-hand side of (4.5) is given by $C_1=\left(\inf_{(x',v')\in\Gamma_-}S(x',v')\inf_{(x,v)\in\Gamma_+}W(x,v)\right)^{-1}$.

We now prove (4.6). Let $x\in\partial X$ be such that $px_0'+(1-p)x\in Z$ for some $p\in(0,1)$. We set $t_0=|x-x_0'|$ and $v_0=\frac{x-x_0'}{|x-x_0'|}$. From (3.16), (3.17), (3.20) and (3.21) it follows that

$$\begin{aligned}&(\tau-|x-x_0'|)^{\frac{3-n}{2}}|\gamma_1-\tilde\gamma_1|(\tau,x,x_0')\le(\tau-|x-x_0'|)^{\frac{3-n}{2}}|\Gamma_2-\tilde\Gamma_2|(\tau,z,z')\\ &+\|(s-|z-z'|)^{\frac{3-n}{2}}(\Gamma_1-\tilde\Gamma_1)(s,z,z')\|_{L^\infty((0,T)_s\times\partial X_z\times\partial X_{z'})},\end{aligned}\tag{6.15}$$

for $\tau>|x-x_0'|$. From (3.17), (3.20)–(3.21) it turns out that $\lim_{\tau\to|x-x_0'|^+}(\tau-|x-x_0'|)^{\frac{3-n}{2}}|\Gamma_2-\tilde\Gamma_2|(\tau,z,z')=0$. Therefore applying (3.26) and (3.29) on the left-hand side of (6.15) we obtain

$$\begin{aligned}&2^{\frac{1-n}{2}}|x-x_0'|^{-\frac{n-1}{2}}C_nS(x_0',v_0)W(x,v_0)|\nu(x_0')\cdot v_0|(\nu(x)\cdot v_0)\\ &\times\left|\int_0^{t_0}\frac{e^{-\int_0^{t_0}\sigma(x_0'+sv_0,v_0)ds}k(x-pv_0,v_0,v_0)-e^{-\int_0^{t_0}\tilde\sigma(x_0'+sv_0,v_0)ds}\tilde k(x-pv_0,v_0,v_0)}{p^{\frac{n-1}{2}}(t_0-p)^{\frac{n-1}{2}}}dp\right|\\ &\le\|(s-|z-z'|)^{\frac{3-n}{2}}(\Gamma_1-\tilde\Gamma_1)(s,z,z'))\|_{L^\infty((0,T)_s\times\partial X_z\times\partial X_{z'})},\end{aligned}\tag{6.16}$$

where $C_n=2$ if $n=2$ and $C_n=\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})$ if $n\ge 3$. Then note that $C_X:=\inf_{x_1\in\partial X,\ z\in\bar Z}\nu(x_1)\cdot\frac{x_1-z}{|x_1-z|}>0$ since X is a bounded convex subset of $\mathbb{R}^n$ with C^1 boundary and $\bar Z\subset X$. Therefore (4.6) follows from (6.16) where the constant C_2 which appears on the right-hand side of (4.6) is given by $C_2=\frac{2^{\frac{n-1}{2}}\mathrm{diam}(X)^{\frac{n-1}{2}}}{C_nC_X^2\inf_{(x',v')\in\Gamma_-}S(x',v')\inf_{(x_1,v_1)\in\Gamma_+}W(x_1,v_1)}$. Theorem 4.2 is proved. □

Proof of Theorem 4.1. We prove (4.1). Let $x_0'\in\partial X$. For $\varepsilon=(\varepsilon_1,\varepsilon_2)\in(0,+\infty)^2$ and $\varepsilon_3\in(0,+\infty)$ let $(f_{\varepsilon_1},g_{\varepsilon_2})\in C^1(\partial X)\times C^1(\mathbb{R})$ satisfy (6.1)–(6.3) and ψ_{ε_3} be defined by (6.4). First note that (6.5)–(6.6) still hold. From Theorems 3.2 and 3.3 it follows that

$$|x-x'|^{n-\frac74}(\tau-|x-x'|)^{\frac34}(\Gamma_1-\tilde\Gamma_1)(\tau,x,x')\in L^\infty((0,T)\times\partial X\times\partial X).\tag{6.17}$$

Combining (6.5), (6.17) and the equality $\|\phi_\varepsilon\|_{L^1((0,\eta)\times\partial X)}$ and the estimate $\|\psi_{\varepsilon_3}\|_{L^\infty((0,T)\times\partial X)} \le 1$ we obtain

$$I_0(\psi_{\varepsilon_3},\phi_\varepsilon) \le \|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T} + C\Delta_2(\psi_{\varepsilon_3},\phi_\varepsilon), \tag{6.18}$$

for $\varepsilon=(\varepsilon_1,\varepsilon_2)\in(0,+\infty)$, $\varepsilon_3\in(0,+\infty)$, where $C=\||x-x'|^{n-\frac{7}{4}}(\tau-|x-x'|)^{\frac{3}{4}}(\Gamma_1-\tilde{\Gamma}_1)(\tau,x,x')\|_{L^\infty((0,T)\times\partial X_x\times\partial X_{x'}}$ and

$$\Delta_2(\psi_{\varepsilon_3},\phi_\varepsilon) = \int_{\substack{(0,T)_t\times\partial X_x\times(0,\eta)_{t'}\times\partial X_{x'}\\ |x-x'|<t-t'}} \frac{\psi_{\varepsilon_3}(t,x)\phi_\varepsilon(t',x')}{|x-x'|^{n-\frac{7}{4}}(t-t'-|x-x'|)^{\frac{3}{4}}}dt d\mu(x) dt' d\mu(x'). \tag{6.19}$$

Note that the function $\Phi_{1,\varepsilon_3}:[0,T)\times\partial X\to\mathbb{R}$ defined by

$$\Phi_{3,\varepsilon_3}(t',x') := \int_{\substack{(0,T)_t\times\partial X_x\\ |x-x'|<t-t'}} \frac{\psi_{\varepsilon_3}(t,x)}{|x-x'|^{n-\frac{7}{4}}(t-t'-|x-x'|)^{\frac{3}{4}}}dt d\mu(x),\ (t',x')\in[0,\eta)\times\partial X, \tag{6.20}$$

is continuous on $[0,\eta)\times\partial X$. Therefore using (6.19), (6.1)–(6.3) and using the equality $\Delta_2(\psi_{\varepsilon_3},\phi_\varepsilon) = \int_{(0,\eta)_{t'}\times\partial X_{x'}}\Phi_{3,\varepsilon_3}(t',x')\phi_\varepsilon(t',x')dt'd\mu(x')$ we obtain

$$\lim_{\varepsilon_3\to 0^+}\lim_{\varepsilon_2\to 0^+}\lim_{\varepsilon_1\to 0^+}\Delta_2(\psi_{\varepsilon_3},\phi_\varepsilon) = \lim_{\varepsilon_3\to 0^+}\Phi_{3,\varepsilon_3}(0,x_0') = 0. \tag{6.21}$$

(we also used (6.20), (6.4) and Lebesgue dominated convergence theorem to prove $\lim_{\varepsilon_3\to 0^+}\Phi_{3,\varepsilon_3}(0,x_0') = 0$). Note that under condition (3.22) the function $\Phi_{4,\varepsilon_3,\varepsilon_2}:\partial X\to\mathbb{R}$ defined by

$$\begin{aligned}\Phi_{4,\varepsilon_3,\varepsilon_2}(x') \;&=\; \int_{\substack{(0,T)_t\times\partial X_x\\ |x-x'|<t}}\psi_{\varepsilon_3}(t,x)g_{\varepsilon_2}(t-|x-x'|)\frac{E_{\partial X\times\partial X}(x,x')-\tilde{E}_{\partial X\times\partial X}(x,x')}{|x-x'|^{n-1}}\\ &\quad\times[W(x,v)S(x',v)(\nu(x)\cdot v)|\nu(x')\cdot v|]_{v=\frac{x-x'}{|x-x'|}}\,dt d\mu(x),\end{aligned} \tag{6.22}$$

is continuous on ∂X. Therefore from (6.1)–(6.3) and the equality $I_0(\psi_{\varepsilon_3},\phi_\varepsilon)=\int_{\partial X} f_{\varepsilon_1}(x')\Phi_{4,\varepsilon_3,\varepsilon_2}(x')d\mu(x')$ (see (6.6)) it follows that

$$\lim_{\varepsilon_1\to 0^+} I_0(\psi_{\varepsilon_3},\phi_\varepsilon) = \Phi_{4,\varepsilon_3,\varepsilon_2}(x_0'),\ \text{for } (\varepsilon_2,\varepsilon_3)\in(0,+\infty). \tag{6.23}$$

Then using Lebesgue dominated convergence theorem and (6.22) we obtain

$$\begin{aligned}\lim_{\varepsilon_3\to 0^+}\lim_{\varepsilon_2\to 0^+}\lim_{\varepsilon_1\to 0^+} I_0(\psi_{\varepsilon_3},\phi_\varepsilon) \;&=\; \int_{\partial X_x}\frac{\left|E_{\partial X\times\partial X}(x,x_0')-\tilde{E}_{\partial X\times\partial X}(x,x_0')\right|}{|x-x_0'|^{n-1}}\\ &\quad\times[W(x,v)S(x_0',v)(\nu(x)\cdot v)|\nu(x_0')\cdot v|]_{v=\frac{x-x_0'}{|x-x_0'|}}\,d\mu(x).\end{aligned} \tag{6.24}$$

Combining (6.24), (6.21) and (6.18) we obtain (4.1).

We prove (4.2)–(4.4). Let $x\in\partial X$ be such that $px_0'+(1-p)x\in X$ for some $p\in(0,1)$. Let $\beta_n:\{(\tau,z,z')\in(0,T)\times\partial X\times\partial X \mid z\neq z',\ \tau>|z-z'|\}\to\mathbb{R}$ be defined by

$$\beta_n(\tau,z,z') = \begin{cases}\sqrt{\tau^2-|z-z'|^2},\ \text{if } n=2,\\ \frac{\tau|z-z'|}{\ln\left(\frac{\tau+|z-z'|}{\tau-|z-z'|}\right)},\ \text{if } n=3,\\ \tau|z-z'|^{n-2},\ \text{if } n\ge 4,\end{cases} \tag{6.25}$$

for $(\tau,z,z')\in(0,T)\times\partial X\times\partial X$, $z\neq z'$, $\tau>|z-z'|$. From (3.13)–(3.15) and (3.17)–(3.19) it follows that

$$\begin{aligned}\beta_n(\tau,x,x_0')|(\gamma_1-\tilde{\gamma}_1)(\tau,x,x_0')| \;&\le\; \beta_n(\tau,x,x_0')|(\Gamma_2-\tilde{\Gamma}_2)(\tau,z,z')|\\ &\quad+\|\beta_n(s,z,z')(\Gamma_1-\tilde{\Gamma}_1)(s,z,z')\|_{L^\infty((0,T)_s\times\partial X_z\times\partial X_{z'})}.\end{aligned} \tag{6.26}$$

From (6.25) and (3.17)–(3.19) it turns out that $\lim_{\tau\to|x-x_0'|^+}\beta_n(\tau,x,x_0')|(\Gamma_2-\tilde\Gamma_2)(\tau,z,z')|=0$. Therefore applying (3.26) (resp. (3.27), (3.28)) on the left-hand side of (6.26) we obtain (4.2) (resp. (4.3), (4.4)). Theorem 4.1 is proved. $\square$

Proof of Theorem 4.5. We first prove (4.11). We extend σ and $\tilde\sigma$ by 0 outside Y. For a bounded and continuous function f on Y consider the X-ray transform $Pf:\mathbb{S}^{n-1}\times\mathbb{R}^n\to\mathbb{R}$ defined by (4.7) (we extend f by 0 outside Y). We recall the following estimate

$$\|f\|_{H^{-\frac12}(Y)}\le\left(\int_{\mathbb{S}^{n-1}}\int_{\Pi_v}|Pf(v,x)|^2dxdv\right)^{\frac12},\tag{6.27}$$

where $\Pi_v:=\{x\in\mathbb{R}^n\mid v\cdot x=0\}$ for $v\in\mathbb{S}^{n-1}$. Note that using the estimate $\|\sigma\|_\infty\le M$, we obtain

$$\int_0^{\tau_+(x_0',v)}\sigma(x_0'+sv,v)ds\le M\tau_+(x_0',v)\le M\mathrm{diam}(X),\ \text{for } (x_0',v)\in\Gamma_-.\tag{6.28}$$

Replacing σ by $\tilde\sigma$ on the left-hand side of (6.28) we obtain an estimate similar to (6.28) for $\tilde\sigma$. Therefore using the estimate $|e^{t_1}-e^{t_2}|\ge e^{-M\mathrm{diam}(X)}|t_1-t_2|$ for $(t_1,t_2)\in[0,+\infty)^2$, $\max(t_1,t_2)\le M\mathrm{diam}(X)$, we obtain

$$\left|e^{-\int_0^{\tau_+(x_0',v)}\sigma(x_0'+sv,v)ds}-e^{-\int_0^{\tau_+(x_0',v)}\tilde\sigma(x_0'+sv,v)ds}\right|\ge e^{-M\mathrm{diam}(X)}\left|P(\sigma-\tilde\sigma)(v,x_0')\right|,\tag{6.29}$$

for $(x_0',v)\in\Gamma_-$. Integrating the left-hand side of (4.5) over ∂X and using (6.29), we obtain

$$\int_{\Gamma_-}\left|P(\sigma-\tilde\sigma)(v,x_0')\right|d\xi(v,x_0')\le e^{M\mathrm{diam}(X)}\mathrm{Vol}(\partial X)C_1\|A_{S,W}-\tilde A_{S,W}\|_{\eta,T},\tag{6.30}$$

where C_1 is the constant that appears on the right-hand side of (4.5). Note that using that X is a convex open subset of $\mathbb{R}^n$ with C^1 boundary we obtain $\int_{\Gamma_-}\left|P(\sigma-\tilde\sigma)(v,x_0')\right|d\xi(v,x_0')=\int_{\mathbb{S}^{n-1}}\int_{\Pi_v}|P(\sigma-\tilde\sigma)(v,x)|dxdv$. Therefore using (6.30) and the estimate $|P(\sigma-\tilde\sigma)(v,x)|^2\le\|\sigma-\tilde\sigma\|_{L^\infty(Y)}\mathrm{diam}(X)|P(\sigma-\tilde\sigma)(v,x)|$ for $(v,x)\in T\mathbb{S}^{n-1}$ (see (6.28) and the estimates $\sigma\ge0$, $\tilde\sigma\ge0$) we obtain

$$\left(\int_{\mathbb{S}^{n-1}}\int_{\Pi_v}|P(\sigma-\tilde\sigma)(v,x)|^2dxdv\right)^{\frac12}\le C_3\|\sigma-\tilde\sigma\|_\infty^{\frac12}\|A_{S,W}-\tilde A_{S,W}\|_{\eta,T}^{\frac12}.\tag{6.31}$$

where $C_3=\left(\mathrm{diam}(X)e^{M\mathrm{diam}(X)}\mathrm{Vol}(\partial X)C_1\right)^{\frac12}$. Combining (6.31) and (6.27) we obtain (4.11).

We now prove (4.14). Let $f\in L^2(X)$, $\mathrm{supp}f\subseteq\bar Z$. We consider the weighted X-ray transform of f, $P_\vartheta f$, defined by

$$P_\vartheta f(x,v)=\int_0^{\tau_+(v,x)}f(pv+x)\vartheta(pv+x,v)dp,\ \text{for a.e. } (x,v)\in\Gamma_-,\tag{6.32}$$

where $\vartheta:X\times\mathbb{S}^{n-1}\to(0,+\infty)$ is the analytic function given by

$$\vartheta(x,v)=(\tau_-(x,v)\tau_+(x,v))^{-\frac{n-1}{2}}g(x,v,v),\ \text{for } (x,v)\in X\times\mathbb{S}^{n-1}.\tag{6.33}$$

From [12, theorem 2.2] and from [16, theorem 4] we obtain

$$\|f\|_{H^{-\frac12}(Z)}\le C\|P_\vartheta f\|_{L^2(\Gamma_-,d\xi)},\tag{6.34}$$

where $C = C(X, Z, g)$ is a constant that does not depend on f. Let $x_0' \in \partial X$ and let $x \in \partial X$ such that $px_0' + (1-p)x \in Z$ for some $p \in (0,1)$ where $v_0 = \frac{x-x_0'}{|x-x_0'|}$ and $t_0 = |x - x_0'|$. Note that using (3.23) (since $\tilde{k} \in L^\infty(Z)$ and $\mathrm{supp}\tilde{k} \subseteq \bar{Z} \subseteq \{x \in X \mid \inf_{x' \in \partial X} |x - x'| \geq \delta\}$), we obtain

$$\begin{aligned}\int_0^{\tau_+(x_0,v_0')} \frac{\tilde{k}(x_0' + pv_0', v_0', v_0')}{p^{\frac{n-1}{2}}(\tau_+(x_0', v_0') - p)^{\frac{n-1}{2}}} dp &\leq \|\tilde{k}\|_{L^\infty(Z)} \int_\delta^{\tau_+(x_0,v_0')-\delta} \frac{1}{p^{\frac{n-1}{2}}(\tau_+(x_0', v_0') - p)^{\frac{n-1}{2}}} dp \\ \leq \|\tilde{k}\|_{L^\infty(Z)} \delta^{-(n-1)} \tau_+(x_0', v_0') &\leq \|\tilde{k}\|_{L^\infty(Z)} \delta^{-(n-1)} \mathrm{diam}(X). \end{aligned} \tag{6.35}$$

We use the estimate

$$\begin{aligned}|P_\vartheta(k_0 - \tilde{k}_0)(x_0', v_0')| \leq\ & e^{P\sigma(v_0',x_0')} |P_\vartheta \tilde{k}_0(x_0', v_0')| \left| e^{-P\sigma(v_0',x_0')} - e^{-P\tilde{\sigma}(v_0',x_0')} \right| \\ &+ e^{P\sigma(v_0',x_0')} \left| e^{-P\sigma(v_0',x_0')} P_\vartheta k_0(x_0', v_0') - e^{-P\tilde{\sigma}(v_0',x_0')} P_\vartheta \tilde{k}_0(x_0', v_0') \right|. \end{aligned} \tag{6.36}$$

Integrating both sides of inequality (6.36) over $v_0' \in \mathbb{S}^{n-1}_{x_0',-}$ and using the estimate $e^{P\sigma(v_0',x_0')} \leq e^{M\mathrm{diam}(X)}$, and using (6.35), (4.5)–(4.6), we obtain

$$\begin{aligned}\int_{\mathbb{S}^{n-1}_{x_0',-}} |P_\vartheta(k_0 - \tilde{k}_0)|(x_0', v_0')|\nu(x_0') \cdot v| dv \leq \delta^{-(n-1)} \mathrm{diam}(X) e^{M\mathrm{diam}(X)} C_1 \|\tilde{k}\|_\infty \|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T} \\ + \frac{\mathrm{Vol}(\mathbb{S}^{n-1}) e^{M\mathrm{diam}(X)} C_2}{2} \left\| (\tau - |z - z'|)^{\frac{n-3}{2}} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \right\|_{L^\infty((0,T)\times\partial X\times\partial X)}, \end{aligned} \tag{6.37}$$

where C_1 and C_2 are the constants that appear on the right-hand side of (4.5) and (4.6).

From the estimate $|P_\vartheta(k_0 - \tilde{k}_0)(v_0', x_0')| \leq \|k - \tilde{k}\|_{L^\infty(Z)} \delta^{-(n-1)} \mathrm{diam}(X)$ for a.e. $(x_0', v_0') \in \Gamma_-$ (see (6.35)), it follows that

$$\|P_\vartheta(k_0 - \tilde{k}_0)\|^2_{L^2(\Gamma_-, d\xi)} \leq \|k - \tilde{k}\|_{L^\infty(Z)} \delta^{-(n-1)} \mathrm{diam}(X) \int_{\partial X} \int_{\mathbb{S}^{n-1}_{x_0',-}} |P_\vartheta(k_0 - \tilde{k}_0)(x_0', v_0')| |\nu(x_0') \cdot v| dv d\mu(x_0'). \tag{6.38}$$

Combining (6.37)–(6.38) and (6.34) we obtain (4.14). □

Proof of Theorem 4.4 (4.13). We first prove (6.42) given below. Note that from (3.24)–(3.25), it follows that

$$|P_{\vartheta_0} f(v, x)| \leq \|f\|_\infty \int_0^{\tau_+(x,v)} \frac{1}{\sqrt{t(\tau_+(x,v) - t)}} dt = C\|f\|_\infty, \tag{6.39}$$

for $(v, x) \in \Gamma_-$ and for $f \in C(X) \cap L^\infty(X)$, where $C = \int_0^1 \frac{1}{\sqrt{t(1-t)}} dt$.

Note that $\nu(x) = x$ and $\nu(x) \cdot (x - x_0') = |\nu(x_0') \cdot (x - x_0')|$ for $(x, x_0') \in \partial X = \mathbb{S}^1$. Therefore from (4.2), it follows that

$$|\nu(x_0') \cdot v_0'|^2 \left| E(x, x_0') P_{\vartheta_0} k_0(v_0', x) - \tilde{E}(x, x_0') P_{\vartheta_0} \tilde{k}_0'(v_0', x) \right| \leq C' \left\| \sqrt{\tau^2 - |z - z'|^2} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \right\|_{L^\infty}, \tag{6.40}$$

for $(x, x_0') \in \partial X^2$, $x \neq x_0'$ and $v_0' = \frac{x-x_0'}{|x-x_0'|}$ (we also used $P_{\vartheta_0} k_{v_0}(v_0, x) = g(v_0, v_0) P_{\vartheta_0} k_0(v_0, x)$ and the similar identity for $\tilde{k}$), where $C' = \frac{1}{2\inf_{\Gamma_-} S \inf_{\Gamma_+} W \inf_{v\in\mathbb{S}^{n-1}} g(v,v)}$. In addition, from (6.40), (6.36) (with P_{ϑ_0} in place of "P_ϑ"), and from (6.28) and (6.39) (with $\tilde{k}_0$ in place of f), it follows that

$$\begin{aligned}|\nu(x_0') \cdot v_0'|^2 |P_{\vartheta_0}(k_0 - \tilde{k}_0)(x_0', v_0')| \leq\ & e^{M\mathrm{diam}(X)} C \|\tilde{k}_0\|_\infty \left| e^{-P\sigma(v_0',x_0')} - e^{-P\tilde{\sigma}(v_0',x_0')} \right| |\nu(x_0') \cdot v_0| \\ &+ e^{M\mathrm{diam}(X)} \left| e^{-P\sigma(v_0',x_0')} P_\vartheta k_0(x_0', v_0') - e^{-P\tilde{\sigma}(v_0',x_0')} P_\vartheta \tilde{k}_0(x_0', v_0') \right| |\nu(x_0') \cdot v_0'|^2. \end{aligned} \tag{6.41}$$

for $(x_0', v_0') \in \Gamma_-$ (we also used the estimate $|\nu(x_0')\cdot v_0'| \le 1$). Performing the change of variables "$x = x_0' + \tau_+(x_0', v_0')v_0'$" ($\frac{\nu(x)\cdot v_0'}{|x-x_0'|^{n-1}}d\mu(x) = dv_0'$) on the left-hand side of (4.1), we obtain that the estimate (4.5) still holds. Using (4.5), (6.41) and (6.40) (and (4.8)), we obtain that there exists a constant C'' such that

$$\begin{aligned}\int_{\mathbb{S}^{n-1}_{x_0'}} |\nu(x_0')\cdot v_0'|^2 |P(\rho(k_0-\tilde k_0))(x_0', v_0')| dv_0' \ \le\ & C''\Big(\|\tilde k_0\|_\infty \|A_{S,W} - \tilde A_{S,W}\|_{\eta,T} \\ & + \Big\|\sqrt{\tau^2 - |z-z'|^2}(\Gamma_1 - \tilde\Gamma_1)(\tau, z, z')\Big\|_{L^\infty}\Big) \quad (6.42)\end{aligned}$$

for $x_0' \in \partial X$. Moreover, using (6.39) (with $k_0 - \tilde k_0$ in place of "f") and Cauchy-Bunyakovski-Schwarz estimate, we obtain

$$\begin{aligned}&\left(\int_{\Gamma_-} |P(\rho(k_0-\tilde k_0))|^2 (v_0, x_0') d\xi(x_0', v_0)\right)^{\frac12} \\ &\le C\|k_0 - \tilde k_0\|_\infty^{\frac34}\left(\int_{\partial X}\int_{\mathbb{S}^1_{x_0',-}} \left(|P(\rho(k_0-\tilde k_0))|(v_0, x_0')|v_0\cdot\nu(x_0')|^2\right)^{\frac12} dv_0' d\mu(x_0')\right)^{\frac12} \\ &\le C\|k_0 - \tilde k_0\|_\infty^{\frac34}\sqrt{2\pi}\left(\int_{\partial X}\int_{\mathbb{S}^1_{x_0',-}} |P(\rho(k_0-\tilde k_0))|(v_0, x_0')|v_0\cdot\nu(x_0')|^2 dv_0' d\mu(x_0')\right)^{\frac14}. \quad (6.43)\end{aligned}$$

Finally combining (6.42)–(6.43), (6.27) (and the identity $\int_{\Gamma_-}\Big|P(\rho(k_0-\tilde k_0)(v, x_0')\Big|^2 d\xi(v, x_0') = \int_{\mathbb{S}^{n-1}}\int_{\Pi_v} |P(k_0-\tilde k_0)(v,x)|^2 dx dv$), we obtain (4.13). $\square$

7 Proof of Theorem 3.2

For $0 < b < a$ we remind that

$$\int_0^{2\pi} \frac{1}{a - b\sin(\Omega)} d\Omega = \frac{2\pi}{\sqrt{a^2-b^2}}. \quad (7.1)$$

We will use the following Lemma 7.1 to prove Theorem 3.2 (3.13), (3.14) and (3.15).

Lemma 7.1. *Let $n \ge 2$. Let N denote the nonnegative measurable function from $(0,T)\times\partial X\times\mathbb{R}^n$ to $[0,+\infty[$ defined by*

$$N(\tau, x, x') = \chi_{(0,+\infty)}(\tau - |x-x'|)\int_{\mathbb{S}^{n-1}} \frac{(\tau - (x-x')\cdot v)^{n-3}}{|x - x' - \tau v|^{2n-4}} dv, \quad (7.2)$$

for $(\tau, x, x') \in (0,T)\times\partial X\times\mathbb{R}^n$. When $n = 2$, then

$$N(\tau, x, x') = \chi_{(0,+\infty)}(\tau - |x-x'|)\frac{2\pi}{\sqrt{\tau^2 - |x-x'|^2}}, \quad (7.3)$$

for $(\tau, x, x') \in (0,T)\times\partial X\times\mathbb{R}^n$. When $n = 3$, then

$$N(\tau, x, x') = 2\pi\frac{\chi_{(0,+\infty)}(\tau - |x-x'|)}{\tau|x-x'|}\ln\left(\frac{\tau + |x-x'|}{\tau - |x-x'|}\right), \quad (7.4)$$

for $(\tau, x, x') \in (0,T)\times\partial X\times\mathbb{R}^n$. When $n \ge 4$, then

$$\sup_{(\tau,x,x')\in(0,T)\times\partial X\times\mathbb{R}^n} \tau|x-x'|^{n-2}N(\tau, x, x') < \infty. \quad (7.5)$$

Proof of Lemma 7.1. Let $(\tau, x, x') \in (0,T) \times \partial X \times \mathbb{R}^n$. We first prove (7.3). Let $n=2$. Note that

$$N(\tau, x, x') = \chi_{(0,+\infty)}(\tau - |x-x'|) \int_0^{2\pi} \frac{1}{\tau - |x-x'|\sin(\Omega)} d\Omega.$$

Therefore using (7.1) we obtain (7.3). We prove (7.4). Let $n=3$. Note that

$$N(\tau, x, x') = 2\pi \frac{\chi_{(0,+\infty)}(\tau - |x-x'|)}{2\tau|x-x'|} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d}{d\Omega} \ln\left(\tau^2 + |x-x'|^2 - 2\tau|x-x'|\sin(\Omega)\right) d\Omega,$$

which gives (7.4).

We prove (7.5). Let $n \geq 4$ and let $(\tau, x, x') \in (0,T) \times \partial X \times \mathbb{R}^n$ be such that $\tau > |x-x'|$ (we remind that $N(\tau, x, x') = 0$ if $\tau \leq |x-x'|$). Using spherical coordinates, we obtain

$$N(\tau, x, x') = \mathrm{Vol}_{n-2}(\mathbb{S}^{n-2}) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\tau - |x-x'|\sin(\Omega))^{n-3}}{(|x-x'|^2 + \tau^2 - 2\tau|x-x'|\sin(\Omega))^{n-2}} \cos(\Omega)^{n-2} d\Omega. \tag{7.6}$$

Performing the change of variables "$r = \frac{\tau^2 - |x-x'|^2}{2(\tau - |x-x'|\sin(\Omega))} - \frac{\tau - |x-x'|}{2}$", we obtain

$$\begin{aligned}
N(\tau, x, x') &= \frac{\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})(\tau^2 - |x-x'|^2)^{\frac{n-3}{2}}}{|x-x'|^{n-2}} \int_0^{|x-x'|} \frac{\sqrt{r(|x-x'|-r)}^{n-3}}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}(\frac{\tau+|x-x'|}{2}-r)^{n-2}} dr \\
&= \frac{2\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})(\tau^2 - |x-x'|^2)^{\frac{n-3}{2}}}{|x-x'|^{n-2}} \int_0^{\frac{|x-x'|}{2}} \frac{\sqrt{r(|x-x'|-r)}^{n-3}}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}(\frac{\tau+|x-x'|}{2}-r)^{n-2}} dr \\
&\leq \frac{2\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})|(\tau^2 - |x-x'|^2)^{\frac{n-3}{2}}}{|x-x'|^{n-2}} \int_0^{\frac{|x-x'|}{2}} \frac{1}{(\frac{\tau-|x-x'|}{2}+r)^{\frac{n-1}{2}}(\frac{\tau+|x-x'|}{2}-r)^{\frac{n-1}{2}}} dr \\
&\leq \frac{2\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})(\tau^2 - |x-x'|^2)^{\frac{n-3}{2}}}{|x-x'|^{n-2}} \frac{\tau}{2}^{\frac{1-n}{2}} \int_0^{\frac{|x-x'|}{2}} \frac{1}{(\frac{\tau-|x-x'|}{2}+r)^{\frac{n-1}{2}}} dr \\
&\leq \frac{2^{n-1}}{n-3} \mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})|x-x'|^{2-n} \left(\frac{\tau+|x-x'|}{2\tau}\right)^{\frac{n-3}{2}} \tau^{-1} \\
&\leq \frac{2^{n-1}}{n-3} \mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})|x-x'|^{2-n}\tau^{-1},
\end{aligned} \tag{7.7}$$

which proves (7.5). $\square$

We are ready to prove Theorem 3.2. First we give an estimate on the simple scattering term. From (3.10) it follows that

$$|\gamma_1(\tau, x, x')| \leq 2^{n-2}\|W\|_\infty \|S\|_\infty \|k\|_\infty I_1(\tau, x, x') \tag{7.8}$$

for a.e. $(\tau, x, x') \in \mathbb{R} \times \partial X \times \partial X$, where

$$\begin{aligned}
I_1(\tau, x, x') &= \chi_{(0,+\infty)}(\tau - |x-x'|) \int_{\mathbb{S}^{n-1}} \chi_{\mathrm{supp}k}(x - sv)_{|s=\frac{\tau^2-|x-x'|^2}{2(\tau - v\cdot(x-x'))}} \\
&\quad \times \frac{(\tau - (x-x')\cdot v)^{n-3}}{|x-x'-\tau v|^{2n-4}} dv.
\end{aligned} \tag{7.9}$$

Let $(\tau, x, x') \in (0,T)\times\partial X\times\partial X$ be such that $x\neq x'$ and $\tau>|x-x'|$. Assume without loss of generality $x'-x=|x'-x|(1,0\ldots 0)$.

First we prove (3.13)–(3.15). From (7.9) and (7.2), it follows that

$$I_1(\tau,x,x')\leq N(\tau,x,x'). \tag{7.10}$$

Combining (7.3) (respectively (7.4), (7.5)) with (7.8) and (7.10), we obtain (3.13) (respectively (3.14), (3.15)).

Now assume that $k\in L^\infty(X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ and

$$\mathrm{supp}k\subseteq\{x\in X\mid \inf_{y\in\partial X}|y-x|\geq\delta\}\text{ for some } 0<\delta<\infty. \tag{7.11}$$

Let $v\in\mathbb{S}^{n-1}$ and $s:=\frac{\tau^2-|x-x'|^2}{2(\tau-(x-x')\cdot v)}$. Straightforward computations give $s+|x-x'-sv|=\tau$. Using (7.11) we obtain that

$$\text{if } \tau<\delta \text{ or } s>\tau-\delta,\text{ then } x-sv\notin\mathrm{supp}k. \tag{7.12}$$

Using (7.9) and (7.12), we obtain

$$\text{if } \tau<\delta \text{ then } I_1(\tau,x,x')=0. \tag{7.13}$$

We prove (3.16) for $n=2$. Using (7.10), we obtain that $I_1(\tau,x,x')\leq\frac{2\pi}{\sqrt{\delta}\sqrt{\tau-|x-x'|}}$ for $\tau\geq\delta$. Combining (7.8) with this latter estimate and (7.13), we obtain (3.16) for $n=2$.

We now prove (3.16) for $n\geq 3$. Let $n\geq 3$ and $\tau\geq\delta$ (the case $\tau<\delta$ is already considered in (7.13)). Performing the change of variables "$r=\frac{\tau^2-|x-x'|^2}{2(\tau-|x-x'|\sin(\Omega))}-\frac{\tau-|x-x'|}{2}$" with "$v=\Phi(\Omega,\omega):=(\sin(\Omega),\cos(\Omega)\omega)$, $\Omega\in(-\frac{\pi}{2},\frac{\pi}{2})$, $\omega\in\mathbb{S}^{n-2}$" on the right-hand side of (7.9), we obtain

$$\begin{aligned}I_1(\tau,x,x') &= 2^{2-n}\frac{(\tau^2-|x-x'|^2)^{\frac{n-3}{2}}}{|x-x'|^{n-2}}\int_0^{|x-x'|}\frac{\sqrt{r(|x-x'|-r)}^{n-3}}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}(\frac{\tau+|x-x'|}{2}-r)^{n-2}}\\ &\quad\int_{\mathbb{S}^{n-2}}\left[\chi_{\mathrm{supp}k}(x-sv)\right]_{\substack{\Omega=\arcsin(|x-x'|^{-1}(\tau-\frac{(\tau^2-|x-x'|^2)}{2(r+\frac{\tau-|x-x'|}{2})}))\\ s=r+\frac{\tau-|x-x'|}{2}\\ v=\Phi(\Omega,\omega)}} d\omega dr.\end{aligned}\tag{7.14}$$

Now assume $\tau>\frac{\delta}{2}+|x-x'|$. Then

$$\begin{aligned}&|x-x'|^{2-n}\int_0^{|x-x'|}\frac{\sqrt{r(|x-x'|-r)}^{n-3}}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}(\frac{\tau+|x-x'|}{2}-r)^{n-2}}dr\\ &\leq\left(\frac{\delta}{4}\right)^{4-2n}|x-x'|^{2-n}\int_0^{|x-x'|}\sqrt{r(|x-x'|-r)}^{n-3}dr=\left(\frac{\delta}{4}\right)^{4-2n}\int_0^1\sqrt{r(1-r)}dr\leq\left(\frac{\delta}{4}\right)^{4-2n}.\end{aligned}$$

Therefore using (7.14) we obtain

$$(\tau-|x-x'|)^{-\frac{n-3}{2}}I_1(\tau,x,x')\leq 2^{n-2}\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})(T+\mathrm{diam}(X))^{\frac{n-3}{2}}\left(\frac{\delta}{2}\right)^{4-2n}. \tag{7.15}$$

Finally assume $\delta\leq\tau\leq\frac{\delta}{2}+|x-x'|$ and $|x-x'|<\tau\leq T$. From (7.12), it follows that

$$(\tau-|x-x'|)^{-\frac{n-3}{2}}I_1(\tau,x,x')\leq\frac{\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})(T+\mathrm{diam}(X))^{\frac{n-3}{2}}}{2^{n-2}|x-x'|^{n-2}}\int_{r_-(\tau,x,x')}^{r_+(\tau,x,x')}\frac{\sqrt{r(|x-x'|-r)}^{n-3}}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}(\frac{\tau+|x-x'|}{2}-r)^{n-2}}dr, \tag{7.16}$$

where

$$r_-(\tau,x,x') := \frac{|x-x'|+\delta-\tau}{2},\ r_+(\tau,x,x') := \frac{\tau-\delta+|x-x'|}{2}. \tag{7.17}$$

Note that

$$\int_{r_-(\tau,x,x')}^{r_+(\tau,x,x')} \frac{\sqrt{r(|x-x'|-r)}^{n-3}}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}(\frac{\tau+|x-x'|}{2}-r)^{n-2}}dr = 2\int_{r_-(\tau,x,x')}^{\frac{|x-x'|}{2}} \frac{\sqrt{r(|x-x'|-r)}^{n-3}}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}(\frac{\tau+|x-x'|}{2}-r)^{n-2}}dr$$

$$\leq 2\left(\frac{\tau}{2}\right)^{2-n}|x-x'|^{n-3}\int_{r_-(\tau,x,x')}^{\frac{|x-x'|}{2}} \frac{1}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}}dr = 2^{n-1}\tau^{2-n}|x-x'|^{n-3}\int_{r_-(\tau,x,x')}^{\frac{|x-x'|}{2}} \frac{1}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}}dr. \tag{7.18}$$

Using (7.17) we obtain

$$\int_{r_-(\tau,x,x')}^{\frac{|x-x'|}{2}} \frac{1}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}}dr = C(n,\tau) := \begin{cases} \ln\left(\frac{\tau}{\delta}\right), \text{ if } n=3, \\ \frac{1}{n-3}\left(\left(\frac{\delta}{2}\right)^{3-n}-\left(\frac{\tau}{2}\right)^{3-n}\right) \text{ otherwise.}\end{cases} \tag{7.19}$$

From (7.16), (7.18), (7.19) and the estimates $\delta \leq \tau < \frac{\delta}{2}+|x-x'|$, it follows that

$$(\tau-|x-x'|)^{-\frac{n-3}{2}}I_1(\tau,x,x') \leq 2^n \mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\delta^{-(n-1)}(T+\mathrm{diam}(X))^{\frac{n-3}{2}}C(n,T), \tag{7.20}$$

where the constant $C(n,T)$ is defined in (7.19). Combining (7.8) with (7.13), (7.15) and (7.20), we obtain (3.16) for $n \geq 3$. □

8 Proof of Theorem 3.3

We shall use the following Lemmas 8.1, 8.2, 8.3 and 8.4. Lemmas 8.1, 8.2, 8.3 and 8.4 are proved in Section 9.

We introduce some notation first. Let $m \geq 1$ and $z', z \in \mathbb{R}^n$ such that $z \neq z'$. Let $\mu \geq 0$. We denote by $\mathcal{E}_{m,n}(\mu,z,z')$ the subset of $(\mathbb{R}^n)^m$ defined by

$$\mathcal{E}_{m,n}(\mu,z,z') = \{(y_1,\ldots,y_m) \in (\mathbb{R}^n)^m \mid |y_1|+\ldots+|y_m|+|z-z'-y_1-\ldots-y_m| < \mu\}. \tag{8.1}$$

When $\mu \leq |z-z'|$, then $\mathcal{E}_{m,n}(\mu,z,z') = \emptyset$.

Lemma 8.1. *Let J_2 be the function from $(0,T)\times\partial X\times\mathbb{R}^n$ defined by*

$$J_2(\mu,z,z') = \int_{\mathcal{E}_{1,n}(\mu,z,z')} \frac{1}{|y|^{n-1}}N(\mu-|y|,z,z'+y)dy, \tag{8.2}$$

where N is defined by (7.2). Then the following statements are valid:

$$\sup_{\substack{(\mu,z,z')\in(0,T)\times\partial X\times\mathbb{R}^n \\ \mu>|z-z'|}} J_2(\mu,z,z') < \infty, \ \textit{when } n=2; \tag{8.3}$$

$$\sup_{\substack{(\mu,z,z')\in(0,T)\times\partial X\times\mathbb{R}^n \\ \mu>|z-z'|}} (\mu-|z-z'|)^{-1}\mu|z-z'|\left(1+\ln\left(\frac{\mu+|z-z'|}{\mu-|z-z'|}\right)\right)^{-2} J_2(\mu,z,z') < \infty, \ \textit{when } n=3; \tag{8.4}$$

$$\sup_{\substack{(\mu,z,z')\in(0,T)\times\partial X\times\mathbb{R}^n \\ \mu>|z-z'|}} (\mu-|z-z'|)^{-1}\mu|z-z'|^{n-2}J_2(\mu,z,z') < \infty, \ \textit{when } n\geq 4. \tag{8.5}$$

Lemma 8.2. *Let $m \geq 3$ and let $\tilde{J}_m$ be the function from $\{(\tau, x, x') \in (0,T) \times \mathbb{R}^n \times \mathbb{R}^n \mid 0 < |x - x'| < \tau\}$ to $\mathbb{R}$ defined by*

$$\tilde{J}_m(\tau, x, x') = \int_{\mathcal{E}_{m-2,n}(\tau,x,x')} \frac{dy_2 \dots dy_{m-1}}{|y_2|^{n-1} \dots |y_{m-1}|^{n-1} |x - x' - y_2 - \dots - y_{m-1}|^{n-2}}, \tag{8.6}$$

for $(\tau, x, x') \in (0,T) \times \mathbb{R}^n \times \mathbb{R}^n$, $0 < |x - x'| < \tau$. When $n = 3$, then there exists a constant $\tilde{C}$ which does not depend on m such that

$$\tilde{J}_m(\tau, x, x') \leq \tilde{C} \frac{\tau - |x - x'|}{|x - x'|} \left(1 + \ln\left(\frac{\tau + |x - x'|}{\tau - |x - x'|}\right)\right) \frac{m^{n-1} \left(\mathrm{Vol}_{n-1}(\mathbb{S}^{n-1})\tau\right)^{m-3}}{(m-3)!}, \tag{8.7}$$

for $(\tau, x, x') \in (0,T) \times \mathbb{R}^n \times \mathbb{R}^n$, $0 < |x - x'| < \tau$. When $n \geq 4$, then there exists a constant $\tilde{C}$ which does not depend on m such that

$$\tilde{J}_m(\tau, x, x') \leq \tilde{C}(\tau - |x - x'|)|x - x'|^{2-n} \frac{m^{n-1} \left(\mathrm{Vol}_{n-1}(\mathbb{S}^{n-1})\tau\right)^{m-3}}{(m-3)!}, \tag{8.8}$$

for $(\tau, x, x') \in (0,T) \times \mathbb{R}^n \times \mathbb{R}^n$, $0 < |x - x'| < \tau$.

Lemma 8.3. *Let $n \geq 2$. Let $(\tau, x, x') \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ be such that $\tau > |x - x'| > 0$, the following estimate is valid:*

$$\mathrm{Vol}_n(\mathcal{E}_{1,n}(\tau, x, x')) \leq \frac{\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\pi(\tau + |x - x'|)}{4} \left(\frac{\sqrt{\tau^2 - |x - x'|^2}}{2}\right)^{n-1}, \tag{8.9}$$

where $\mathcal{E}_{1,n}$ is defined by (8.1).

Lemma 8.4. *Let B be the function from $\{(\mu, z, z') \in (0,T) \times \partial X \times \mathbb{R}^3 \mid \mu > |z - z'| > 0\}$ to $\mathbb{R}$ defined by*

$$B(\mu, z, z') := \int_{\mathcal{E}_{1,3}(\mu,z,z')} \ln\left(\frac{\mu - |y| + |z - z' - y|}{\mu - |y| - |z - z' - y|}\right) dy = \int_{\mathcal{E}_{1,3}(\mu,t_0(1,0,0),0)} \ln\left(\frac{\mu - |y| + |(t_0, 0, 0) - y|}{\mu - |y| - |(t_0, 0, 0) - y|}\right) dy, \tag{8.10}$$

for $(\mu, z, z') \in (0,T) \times \partial X \times \mathbb{R}^n$ where $t_0 = |z - z'|$, $z \neq z'$, $\mu > |z - z'|$. Then we have:

$$\sup_{\substack{(\mu,z,z') \in (0,T) \times \partial X \times \mathbb{R}^3 \\ \mu > |z - z'| > 0}} (\mu - |z - z'|)^{-1} \left(1 + \ln\left(\frac{\mu + |z - z'|}{\mu - |z - z'|}\right)\right)^{-1} B(\mu, z, z') < \infty. \tag{8.11}$$

We also need the explicit expression of γ_m, $m \geq 2$, to prove Theorem 3.3

$$\begin{aligned}
\gamma_2(\tau, x, x') := &\int_{\substack{y \in \mathcal{E}_{1,n}(\tau,x,x') \\ x'+y \in X}} \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x) \cdot v) W(x, v) \left[E(x, x - (\tau - |y| - s_1)v, x' + y, x')\right. \\
&\chi_{(0,\tau_-(x,v))}(\tau - |y| - s_1) k(x - (\tau - s_1 - |y|)v, v_1, v) k(x' + y, v', v_1) S(x', v') \\
&|\nu(x') \cdot v'|\Big]_{\substack{s_1 = \frac{|x - x' - y - (\tau - |y|)v|^2}{2(\tau - |y| - (x - x' - y) \cdot v)} \\ v_1 = \frac{x - x' - y - (\tau - s_1 - |y|)v}{s_1} \\ v' = \frac{y}{|y|}}} \frac{2^{n-2}(\tau - |y| - (x - x' - y) \cdot v)^{n-3}}{|x - x' - y - (\tau - |y|)v|^{2n-4} |y|^{n-1}} dy dv,
\end{aligned} \tag{8.12}$$

and

$$
\begin{aligned}
\gamma_m(\tau,x,x') :=& \int_{\substack{(y_2,\ldots,y_m)\in\mathcal{E}_{m-1,n}(\tau,x,x')\\(x'+y_m,\ldots,x'+y_m+\ldots+y_2)\in X^{m-1}}} \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x)\cdot v)W(x,v)\\
&\times\frac{2^{n-2}\left(\tau-|y_2|-\ldots-|y_m|-(x-x'-y_2-\ldots-y_m)\cdot v\right)^{n-3}}{|y_2|^{n-1}\ldots|y_m|^{n-1}|x-x'-y_2-\ldots-y_m-(\tau-|y_2|-\ldots-|y_m|)v|^{2n-4}}\\
&\times\big[\chi_{(0,\tau_-(x,v))}(\tau-s_1-|y_2|-\ldots-|y_m|)E(x,x-(\tau-s_1-|y_2|-\ldots-|y_m|)v,\\
&x'+y_m\ldots+y_2,\ldots,x'+y_m,x')k(x-(\tau-s_1-|y_2|-\ldots-|y_m|)v,v_1,v)\\
&\times k(x'+y_m+\ldots+y_2,v_2,v_1)\ldots k(x'+y_m+\ldots+y_{i+1},v_{i+1},v_i)\ldots\\
&k(x'+y_m+y_{m-1},v_{m-1},v_{m-2})k(x'+y_m,v',v_{m-1})S(x',v')\\
&|\nu(x')\cdot v'|\big]_{\substack{v_1=\frac{x-x'-y_2-\ldots-y_m-(\tau-s_1-|y_2|-\ldots-|y_m|)v}{s_1}\\ s_1=\frac{|x-x'-y_2-\ldots-y_m-(\tau-|y_2|-\ldots-|y_m|)v|^2}{2(t-|y_2|-\ldots-|y_{m-1}|-(x-x'-y_2-\ldots y_{m-1})\cdot v)}\\ v'=\frac{y_m}{|y_m|}\\ v_i=\frac{y_i}{|y_i|},\ i=2\ldots m-1}} dy_2\ldots dy_m dv,
\end{aligned}
\tag{8.13}
$$

for $\tau\in\mathbb{R}$ and a.e. $(x,x')\in\partial X\times\partial X$ and for $m\geq 3$.

We are ready to prove Theorem 3.3. We prove (3.17), (3.18) and (3.19). Let $\tau\in(0,T)$ and let $x\in\partial X$, $x'\in\partial X$ and $x\neq x'$. Set $t_0=|x-x'|$. We first look for an upper bound on $|\gamma_2(\tau,x,x')|$. Using (8.12) and the fact that σ is a nonnegative function, we obtain

$$
|\gamma_2(\tau,x,x')|\leq 2^{n-2}\|W\|_\infty\|S\|_\infty\|k\|_\infty^2 J_2(\tau,x,x'),\tag{8.14}
$$

where J_2 and $\mathcal{E}_{1,n}(\tau,x,x')$ are defined by (8.2) and (8.1). From (8.14) and (8.3)–(8.5) it follows that there exists a real constant C such that

$$
|\gamma_2(\tau,x,x')|\leq C\|W\|_\infty\|S\|_\infty\|k\|_\infty^2 \sup_{\substack{(s,z,z')\in(0,T)\times\mathbb{R}^n\times\mathbb{R}^n\\ s>|z-z'|}} J_2(s,z,z'),\ \text{when } n=2,\tag{8.15}
$$

and

$$
\frac{\tau|x-x'|}{(\tau-|x-x'|)\left(1+\ln\left(\frac{\tau+|x-x'|}{\tau-|x-x'|}\right)\right)^2}|\gamma_2(\tau,x,x')|\leq C\|W\|_\infty\|S\|_\infty\|k\|_\infty^2,\ \text{when } n=3,\tag{8.16}
$$

and

$$
\frac{\tau|x-x'|^{n-2}}{\tau-|x-x'|}|\gamma_2(\tau,x,x')|\leq C\|W\|_\infty\|S\|_\infty\|k\|_\infty^2,\ \text{when } n\geq 4.\tag{8.17}
$$

Let $m\geq 3$. Using (8.13) we obtain

$$
|\gamma_m(\tau,x,x')|\leq 2^{n-2}\|W\|_\infty\|S\|_\infty\|k\|_\infty^m J_m(\tau,x,x'),\tag{8.18}
$$

where

$$
J_m(\tau,x,x')=\int_{\mathcal{E}_{m-2,n}(\tau,x,x')}\frac{J_2(\tau(\bar y),w(\bar y),0)}{|y_2|^{n-1}\ldots|y_{m-1}|^{n-1}}dy_2\ldots dy_{m-1},\tag{8.19}
$$

and J_2 (resp. $\mathcal{E}_{m-2,n}(\tau,x,x')$) is defined by (8.2) (resp. (8.1)) and where $\bar y=(y_2,\ldots,y_{m-1})$, $\tau(\bar y)=\tau-|y_2|-\ldots-|y_{m-1}|$, $t_0(\bar y)=|x-x'-y_2-\ldots-y_{m-1}|$ and $w(\bar y)=x-x'-y_2-\ldots-y_{m-1}$ for $\bar y\in(\mathbb{R}^n)^{m-2}$.

Assume $n = 2$. Then using (8.19), (8.3) and spherical coordinates (and (8.1)), we obtain

$$\begin{aligned}
J_m(\tau, x, x') &\leq \sup_{\substack{(s,z,z')\in(0,T)\times\mathbb{R}^n\times\mathbb{R}^n \\ s>|z-z'|}} J_2(s, z, z') \int_{\mathcal{E}_{m-2,n}(\tau,x,x')} \frac{1}{|y_2|\ldots|y_{m-1}|} d\bar{y} \\
&\leq (2\pi)^{m-2} \sup_{\substack{(s,z,z')\in(0,T)\times\mathbb{R}^n\times\mathbb{R}^n \\ s>|z-z'|}} J_2(s, z, z') \int_{\substack{s_2+\ldots+s_{m-1}\leq\tau \\ s_i\geq 0,\ i=2\ldots m-1}} ds_2 \ldots ds_{m-1} \\
&= (2\pi)^{m-2} \frac{\tau^{m-2}}{(m-2)!} \sup_{\substack{(s,z,z')\in(0,T)\times\mathbb{R}^n\times\mathbb{R}^n \\ s>|z-z'|}} J_2(s, z, z'). \qquad (8.20)
\end{aligned}$$

Finally combining (8.20) and (8.18), we obtain

$$|\gamma_m(\tau, x, x')| \leq (2\pi)^{m-2} \|W\|_\infty \|S\|_\infty \|k\|_\infty^m \frac{\tau^{m-2}}{(m-2)!} \sup_{\substack{(s,z,z')\in(0,T)\times\mathbb{R}^n\times\mathbb{R}^n \\ s>|z-z'|}} J_2(s, z, z'), \qquad (8.21)$$

Statement (3.17) follows from (8.15), (8.21).

Assume $n \geq 3$. Note that

$$\frac{\mu - |z - z'|}{\mu} = 1 - \frac{|z - z'|}{\mu} \leq 1, \qquad (8.22)$$

and

$$\frac{\mu - |z - z'|}{\mu} \left(1 + \ln\left(\frac{\mu + |z - z'|}{\mu - |z - z'|}\right)\right)^2 \leq \sup_{s\in(0,1)} (1 - s) \left(1 + \ln\left(\frac{1 + s}{1 - s}\right)\right)^2, \qquad (8.23)$$

for $(\mu, z, z') \in (0, T) \times \mathbb{R}^n \times \mathbb{R}^n$ such that $|z - z'| < \mu$.

From (8.19), (8.22) and (8.4), (8.23) and (8.5), it follows that there exists a real constant C such that

$$J_m(\tau, x, x') \leq C \tilde{J}_m(\tau, x, x'), \qquad (8.24)$$

where $\tilde{J}_m$ is defined by (8.6).

Assume $n = 3$. Combining (8.18), (8.24), (8.7), we obtain that there exists a real constant C' (which does not depend on τ, x, x' and m) such that

$$|\gamma_m(\tau, x, x')| \leq C' \|W\|_\infty \|S\|_\infty \|k\|_\infty^m \frac{\tau - |x - x'|}{|x - x'|} \left(1 + \ln\left(\frac{\tau + |x - x'|}{\tau - |x - x'|}\right)\right) \frac{m^{n-1} \left(\mathrm{Vol}_{n-1}(\mathbb{S}^{n-1})\tau\right)^{m-3}}{(m-3)!}. \qquad (8.25)$$

Statement (3.18) follows from (8.16) and (8.25).

Now assume $n \geq 4$. Combining (8.18), (8.24), (8.8), we obtain that there exists a real constant C' (which does not depend on τ, x, x' and m) such that

$$|\gamma_m(\tau, x, x')| \leq C' \|W\|_\infty \|S\|_\infty \|k\|_\infty^m (\tau - |x - x'|) |x - x'|^{2-n} \frac{m^{n-1} \left(\mathrm{Vol}_{n-1}(\mathbb{S}^{n-1})\tau\right)^{m-3}}{(m-3)!}. \qquad (8.26)$$

Statement (3.19) follows from (8.17) and (8.26).

We now prove (3.20)–(3.21). Let $n \geq 3$ and $m \geq 2$. From the expression of γ_m (see (8.12)–(8.13)), it follows that

$$|\gamma_m(\tau, x, x')| \leq 2^{n-2} \|W\|_\infty \|S\|_\infty \|k\|_\infty^m I_m(\tau, x, x') \qquad (8.27)$$

where

$$I_m(\tau,x,x') := \int\limits_{\substack{(y_2,\dots,y_m)\in\mathcal{E}_{m-1,n}(\tau,x,x'_0)\\ (x'+y_m,\dots,x'+\sum_{i=2}^m y_i)\in(\mathrm{supp}k)^{m-1}}} \frac{N(\tau-\sum_{i=2}^m|y_i|,x,x'+\sum_{i=2}^m y_i)dy_m\dots dy_2}{|y_2|^{n-1}\dots|y_m|^{n-1}}, \quad (8.28)$$

where N and $\mathcal{E}_{m-1,n}(\tau,x,x')$ are defined by (7.2) and (8.1). Note that

$$|y_m|\ge\delta \quad \text{and} \quad \tau-\sum_{i=2}^m|y_i|\ge|x-x'-y_2-\dots-y_m|\ge\delta, \quad (8.29)$$

for $(y_2,..,y_m)\in\mathcal{E}_{m-1,n}(\tau,x,x')$ such that $x'+y_m\in\mathrm{supp}k$ and $x'+y_2+\dots+y_m\in\mathrm{supp}k$ since $\mathrm{supp}k\subseteq\{z\in X \mid \inf_{y\in\partial X}|y-z|\ge\delta\}$.

We prove (3.20). Assume $n=3$. Using (8.28)–(8.29), (7.4) and (8.10) we obtain

$$I_2(\tau,x,x')\le 2\pi\delta^{-4}B(\tau,x,x'). \quad (8.30)$$

Therefore using (8.11) and (8.30) we obtain

$$\sup_{\substack{(s,z,z')\in(0,T)\times\partial X\times\partial X\\ s>|z-z'|>0}}(s-|z-z'|)^{-1}\left(1+\ln\left(\frac{s+|z-z'|}{s-|z-z'|}\right)\right)^{-1}I_2(s,z,z')<\infty. \quad (8.31)$$

Now assume $m\ge3$. Using (8.28), we obtain

$$I_m(\tau,x,x')\le\int\limits_{\substack{(y_3,\dots,y_m)\in\mathcal{E}_{m-2,n}(\tau,x,x'_0)\\ (x'+\sum_{i=3}^m y_i,x'+y_m)\in(\mathrm{supp}k)^2}}\frac{J_2(\tau-\sum_{i=3}^m|y_i|,x,x'+\sum_{i=3}^m y_i)dy_m\dots dy_3}{|y_3|^{n-1}\dots|y_m|^{n-1}}, \quad (8.32)$$

where J_2 is defined by (8.2). Using (8.32) and (8.4) and the estimate $\sup_{r\in(0,1)}r(1-\ln(r))^2<\infty$ we obtain

$$I_m(\tau,x,x')\le D\int\limits_{\substack{(y_3,\dots,y_m)\in\mathcal{E}_{m-2,n}(\tau,x,x'_0)\\ (x'+\sum_{i=3}^m y_i,x'+y_m)\in(\mathrm{supp}k)^2}}\frac{(|x-x'-y_3-\dots-y_m|+\tau-\sum_{i=3}^m|y_i|)dy_m\dots dy_3}{|y_3|^{n-1}\dots|y_m|^{n-1}(\tau-\sum_{i=3}^m|y_i|)|x-x'-y_3-\dots-y_m|}, \quad (8.33)$$

where $D:=\sup_{r\in(0,1)}r(1-\ln(r))^2\sup_{\substack{(s,z,z')\in(0,T)\times\partial X\times\partial X\\ s>|z-z'|>0}}(s-|z-z'|)^{-1}s|z-z'|\left(1+\ln\left(\frac{s+|z-z'|}{s-|z-z'|}\right)\right)^{-2}$ $J_2(s,z,z')$. If $m=3$, then using (8.29) with "$(y_2,\dots,y_m)$" replaced by "$(y_3,\dots,y_m)$", we obtain

$$I_3(\tau,x,x')\le 2\tau\delta^{-4}D\mathrm{Vol}(\mathcal{E}_{1,3}(\tau,x,x')) \quad (8.34)$$

(we also used the estimate $|x-x'-y_3|+\tau-|y_3|\le 2\tau$ for $y_3\in\mathcal{E}_{1,3}(\tau,x,x')$). If $m\ge4$, then using (8.29) with "$(y_2,\dots,y_m)$" replaced by "$(y_3,\dots,y_m)$", we obtain

$$\begin{aligned}
I_m(\tau,x,x') &\le 2\tau\delta^{-4}D\mathrm{Vol}(\mathcal{E}_{1,3}(\tau,x,x'))\int\limits_{(y_3,\dots,y_{m-1})\in\mathcal{E}_{m-3,3}(\tau,x,x')}\frac{dy_3\dots dy_{m-1}}{|y_3|^{n-1}\dots|y_{m-1}|^{n-1}}\\
&\le 2\tau\delta^{-4}D\mathrm{Vol}(\mathcal{E}_{1,3}(\tau,x,x'))\mathrm{Vol}(\mathbb{S}^{n-1})^{m-3}\int_{\substack{(s_3,\dots,s_{m-1})\in(0,+\infty)^{m-3}\\ s_3+\dots+s_{m-1}<\tau}}ds_3\dots ds_{m-1}\\
&= 2\tau\delta^{-4}D\mathrm{Vol}(\mathbb{S}^{n-1})^{m-3}\frac{\tau^{m-3}}{(m-3)!}\mathrm{Vol}(\mathcal{E}_{1,3}(\tau,x,x') \quad (8.35)
\end{aligned}$$

(we also used the estimate $|x-x'-y_3-\ldots-y_m|+\tau-|y_3|-\ldots-|y_m|\le 2\tau$ for $(y_3,\ldots,y_m)\in\mathcal{E}_{m-2,3}(\tau,x,x')$ and we performed the changes of variables $y_i=s_i\omega_i$, $(s_i,\omega_i)\in(0,+\infty)\times\mathbb{S}^{n-1}$). Statement (3.20) follows from (8.27), (8.31) and (8.34)–(8.35) (and (8.9)).

We prove (3.21). Let $n\ge 4$. Using (8.28) and (7.5), we obtain

$$I_m(\tau,x,x')\le \||s|z-z'|^{n-2}N(s,z,z')\|_{L^\infty(\mathbb{R}_s\times\partial X_z\times\mathbb{R}^n_{z'})}$$
$$\times\int\limits_{\substack{(y_2,\ldots,y_m)\in\mathcal{E}_{m-1,n}(\tau,x,x'_0)\\ x'+y_m\in\mathrm{supp}k\\ x'+\sum_{i=2}^m y_i\in\mathrm{supp}k}}\frac{dy_m\ldots dy_2}{|y_2|^{n-1}\ldots|y_m|^{n-1}|x-x'-\sum_{i=2}^m y_i|^{n-2}(\tau-\sum_{i=2}^m|y_i|)}. \quad (8.36)$$

Assume $m=2$. Using (8.29) and (8.36), we obtain

$$I_m(\tau,x,x')\le\delta^{-2n+2}\||s|z-z'|^{n-2}N(s,z,z')\|_{L^\infty(\mathbb{R}_s\times\partial X_z\times\mathbb{R}^n_{z'})}\mathrm{Vol}(\mathcal{E}_{1,n}(\tau,x,x')). \quad (8.37)$$

Therefore using (8.9), we obtain

$$I_m(\tau,x,x')\le\delta^{-2n+2}\||s|z-z'|^{n-2}N(s,z,z')\|_{L^\infty(\mathbb{R}_s\times\partial X_z\times\mathbb{R}^n_{z'})}\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\pi(\tau+|x-x'|)\left(\frac{\sqrt{\tau^2-|x-x'|^2}}{2}\right)^{n-1}. \quad (8.38)$$

Assume $m\ge 3$. Using (8.29) and (8.36), we obtain

$$I_m(\tau,x,x')\le\delta^{-2n+2}\||s|z-z'|^{n-2}N(s,z,z')\|_{L^\infty(\mathbb{R}_s\times\partial X_z\times\mathbb{R}^n_{z'})}\int\limits_{(y_2,\ldots,y_m)\in\mathcal{E}_{m-1,n}(\tau,x,x')}\frac{dy_m\ldots dy_2}{|y_2|^{n-1}\ldots|y_{m-1}|^{n-1}}. \quad (8.39)$$

Note that $|y_m|+|x-x'-y_m|\le|y_2|+\ldots+|y_m|+|x-x'-y_2-\ldots-y_m|$ for $(y_2,\ldots,y_m)\in(\mathbb{R}^n)^{m-1}$. Hence

$$|y_2|+\ldots+|y_{m-1}|<\tau-|y_m| \qquad \text{and} \qquad |y_m|+|x-x'-y_m|<\tau, \quad (8.40)$$

for $(y_2,\ldots,y_m)\in\mathcal{E}_{m-1,n}(\tau,x,x')$ (see (8.1)). Therefore

$$\begin{aligned}
&I_m(\tau,x,x')\le\delta^{-2n+2}\||s|z-z'|^{n-2}N(s,z,z')\|_{L^\infty(\mathbb{R}_s\times\partial X_z\times\mathbb{R}^n_{z'})}\\
&\times\int_{y_m\in\mathcal{E}_{1,n}(\tau,x,x')}\int_{\sum_{i=2}^{m-1}|y_i|<\tau-|y_m|}\frac{dy_m\ldots dy_2}{|y_2|^{n-1}\ldots|y_{m-1}|^{n-1}}\\
&=\delta^{-2n+2}\mathrm{Vol}(\mathbb{S}^{n-1})^{m-2}\||s|z-z'|^{n-2}N(s,z,z')\|_{L^\infty(\mathbb{R}_s\times\partial X_z\times\mathbb{R}^n_{z'})}\int_{y_m\in\mathcal{E}_{1,n}(\tau,x,x')}\frac{(\tau-|y_m|)^{m-2}}{(m-2)!}dy_m\\
&\le\delta^{-2n+2}\mathrm{Vol}(\mathbb{S}^{n-1})^{m-2}\||s|z-z'|^{n-2}N(s,z,z')\|_{L^\infty(\mathbb{R}_s\times\partial X_z\times\mathbb{R}^n_{z'})}\mathrm{Vol}(\mathcal{E}_{1,n}(\tau,x,x'))\frac{\tau^{m-2}}{(m-2)!}\\
&=\delta^{-2n+2}\mathrm{Vol}(\mathbb{S}^{n-1})^{m-2}\||s|z-z'|^{n-2}N(s,z,z')\|_{L^\infty(\mathbb{R}_s\times\partial X_z\times\mathbb{R}^n_{z'})}\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\pi(\tau+|x-x'|)\\
&\times\left(\frac{\sqrt{\tau^2-|x-x'|^2}}{2}\right)^{n-1}\frac{\tau^{m-2}}{(m-2)!}. \quad (8.41)
\end{aligned}$$

Statement (3.21) follows from (8.27), (8.38) and (8.41). $\square$

9 Proof of Lemmas 8.1, 8.2, 8.3 and 8.4

We remind the following change of variables for the proof of Lemmas 8.1, 8.2, 8.3 and 8.4.

$$\int_{\mathcal{E}_{1,n}(\tau,t_0v,0))} f(y)dy = \begin{cases} \int_{(0,2\pi)\times(t_0,\tau)} f\left(\frac{t_0+s\cos(\varphi)}{2}, \frac{\sqrt{s^2-t_0^2}}{2}\sin\varphi\right) \\ \times\frac{(s^2-t_0^2\cos^2(\varphi))}{4\sqrt{s^2-t_0^2}}dsd\varphi, \text{ if } n=2, \\ \int_{\mathbb{S}^{n-2}\times(0,\pi)\times(t_0,\tau)} f\left(\frac{t_0+s\cos(\varphi)}{2}, \frac{\sqrt{s^2-t_0^2}}{2}\sin\varphi\omega\right) \\ \times\left(\frac{\sin(\varphi)\sqrt{s^2-t_0^2}}{2}\right)^{n-2}\frac{s^2-t_0^2\cos^2(\varphi)}{4\sqrt{s^2-t_0^2}}d\omega dsd\varphi, \text{ if } n\geq 3, \end{cases} \tag{9.1}$$

for $f\in L^1(\mathbb{R}^n)$ and $(\tau,t_0,v)\in(0,+\infty)\times(0,+\infty)\times\mathbb{S}^{n-1}$ such that $\tau>t_0$.

Proof of Lemma 8.1. We prove (8.3). Let $(\mu,z,z')\in(0,T)\times\partial X\times\mathbb{R}^2$ be such that $\mu>|z-z'|$. From (8.2) and (7.3) it follows that

$$\begin{aligned} J_2(\mu,z,z') &= \int_{\mathcal{E}_{1,2}(\mu,z,z')}\frac{2\pi}{|y|\sqrt{(\mu-|y|)^2-|z-z'-y|^2}}dy \\ &= \int_{\mathcal{E}_{1,2}(\mu,t_0(1,0),0)}\frac{2\pi}{|y|\sqrt{(\mu-|y|)^2-|t_0(1,0)-y|^2}}dy, \end{aligned} \tag{9.2}$$

where $t_0=|z-z'|$.

Using the change of variables $y=\frac{t_0}{2}(1,0)+(s\cos(\varphi),\frac{\sqrt{s^2-t_0^2}}{2}\sin(\varphi))$ (see (9.1)), $\varphi\in(0,2\pi)$, $s\in(t_0,\mu)$, we obtain

$$J_2(\mu,z,z')=4\pi\int_{t_0}^{\mu}\int_0^{2\pi}J_{2,1}(\mu,s,\varphi)d\varphi ds, \tag{9.3}$$

where

$$J_{2,1}(\mu,s,\varphi)=\frac{s-t_0\cos(\varphi)}{\sqrt{s^2-t_0^2}\sqrt{\mu-s}\sqrt{\mu-t_0\cos(\varphi)}}, \tag{9.4}$$

for $\varphi\in(0,2\pi)$ and $s\in(t_0,\mu)$.

We give an estimate on $J_{2,1}$. From (9.4) and the estimates $\mu-t_0\cos(\varphi)\geq s-t_0\cos(\varphi)$, $s+t_0\geq s-t_0\cos(\varphi)$, it follows that

$$J_{2,1}(\mu,s,\varphi)\leq\frac{1}{\sqrt{s-t_0}\sqrt{\mu-s}}, \tag{9.5}$$

for $\varphi\in(0,\frac{\pi}{2})$ and $s\in(t_0,\mu)$. Performing the change of variables $s=t_0+\varepsilon(\mu-t_0)$ we have

$$\int_{t_0}^{\mu}\frac{1}{\sqrt{s-t_0}\sqrt{\mu-s}}ds=\int_0^1\frac{1}{\sqrt{\varepsilon(1-\varepsilon)}}d\varepsilon<+\infty, \tag{9.6}$$

for $s\in(t_0,\mu)$. Combining (9.3), (9.5), (9.6), we obtain

$$\sup_{\substack{(\mu,z,z')\in(0,T)\times\partial X\times\mathbb{R}^2\\ \mu>|z-z'|}}J_2(\mu,z,z')\leq 8\pi^2\int_0^1\frac{1}{\sqrt{\varepsilon(1-\varepsilon)}}d\varepsilon<\infty. \tag{9.7}$$

Statement (8.3) follows from (9.7).

We prove (8.4). Let $(\mu, z, z') \in (0,T)\times\partial X\times\mathbb{R}^3$ be such that $\mu > |z-z'|$. Set $t_0 = |z-z'|$. From (8.2), (7.4) and (9.1), it follows that

$$\begin{aligned} J_2(\mu,z,z') &\le \int_{\mathcal{E}_{1,3}(\mu,z,z')} \frac{2\pi\ln\left(\frac{\mu-|y|+|z-z'-y|}{\mu-|y|-|z-z'-y|}\right)}{|y|^2(\mu-|y|)|z-z'-y|}dy \\ &= \int_{\mathcal{E}_{1,3}(\mu,t_0(1,0,0),0)} \frac{2\pi\ln\left(\frac{\mu-|y|+|t_0(1,0,0)-y|}{\mu-|y|-|t_0(1,0,0)-y|}\right)}{|y|^2(\mu-|y|)|t_0(1,0,0)-y|}dy \\ &= 8\pi^2\int_{t_0}^{\mu}\int_0^{\pi} J_{2,1}(\mu,s,\varphi)d\varphi ds, \end{aligned} \tag{9.8}$$

where

$$\begin{aligned} J_{2,1}(\mu,s,\varphi) &= \frac{\sin(\varphi)\ln\left(\frac{\mu-t_0\cos(\varphi)}{\mu-s}\right)}{(s+t_0\cos(\varphi))(2\mu-s-t_0\cos(\varphi))} \\ &= \ln\left(\frac{\mu-t_0\cos(\varphi)}{\mu-s}\right)\left(\frac{\sin(\varphi)}{2\mu(s+t_0\cos(\varphi))}+\frac{\sin(\varphi)}{2\mu(2\mu-s-t_0\cos(\varphi))}\right), \end{aligned} \tag{9.9}$$

for $\varphi\in(0,\pi)$ and $s\in(t_0,\mu)$. From (9.9) and the estimates $2\mu-s-t_0\cos(\varphi)\ge\mu-t_0\cos(\varphi)$, $0\le\ln\left(\frac{\mu-t_0\cos(\varphi)}{\mu-s}\right)\le\ln\left(\frac{\mu+t_0}{\mu-s}\right)$, it follows that

$$J_{2,1}(\mu,s,\varphi)\le\ln\left(\frac{\mu+t_0}{\mu-s}\right)\left(\frac{\sin(\varphi)}{2\mu(s-t_0\cos(\varphi))}+\frac{\sin(\varphi)}{2\mu(\mu-t_0\cos(\varphi))}\right),$$

for $\varphi\in(0,\pi)$ and $s\in(t_0,\mu)$. Therefore

$$\begin{aligned} \int_0^{\pi} J_{2,1}(\mu,s,\varphi)d\varphi &\le \frac{\ln\left(\frac{\mu+t_0}{\mu-s}\right)}{2\mu t_0}\left(\ln\left(\frac{s+t_0}{s-t_0}\right)+\ln\left(\frac{\mu+t_0}{\mu-t_0}\right)\right) \\ &\le \frac{\ln\left(\frac{\mu+t_0}{\mu-s}\right)}{2\mu t_0}\left(\ln\left(\frac{\mu+t_0}{s-t_0}\right)+\ln\left(\frac{\mu+t_0}{\mu-t_0}\right)\right). \end{aligned} \tag{9.10}$$

We remind the following integral value

$$\int_{t_0}^{\mu}\ln\left(\frac{\mu+t_0}{\mu-s}\right)ds = (\mu-t_0)\ln\left(\frac{\mu+t_0}{\mu-t_0}\right)+\mu-t_0. \tag{9.11}$$

Using the estimate $\ln\left(\frac{\mu+t_0}{\mu-s}\right)\le\ln\left(\frac{2(\mu+t_0)}{\mu-t_0}\right)$ for $s\in(t_0,\frac{t_0+\mu}{2})$, we obtain

$$\begin{aligned} &\int_{t_0}^{\mu}\ln\left(\frac{\mu+t_0}{s-t_0}\right)\ln\left(\frac{\mu+t_0}{\mu-s}\right)ds = 2\int_{t_0}^{\frac{t_0+\mu}{2}}\ln\left(\frac{\mu+t_0}{s-t_0}\right)\ln\left(\frac{\mu+t_0}{\mu-s}\right)ds \\ &\le 2\ln\left(\frac{2(\mu+t_0)}{\mu-t_0}\right)\int_{t_0}^{\frac{t_0+\mu}{2}}\ln\left(\frac{\mu+t_0}{s-t_0}\right)ds \\ &\le 2(\mu-t_0)\left(\ln\left(\frac{(\mu+t_0)}{\mu-t_0}\right)+\ln(2)\right)\left(\ln\left(\frac{\mu+t_0}{\mu-t_0}\right)+1\right). \end{aligned} \tag{9.12}$$

Combining (9.8)–(9.12) and (9.11), we obtain

$$J_2(\mu,z,z') \le 4\pi^2 \frac{\mu-t_0}{\mu t_0}\left(3\ln\left(\frac{(\mu+t_0)}{\mu-t_0}\right)+2\ln(2)\right)\left(\ln\left(\frac{\mu+t_0}{\mu-t_0}\right)+1\right). \tag{9.13}$$

Statement (8.4) follows from (9.13).

We prove (8.5). Let $n\ge 4$. Let $(\mu,z,z')\in(0,T)\times\partial X\times\mathbb{R}^n$ be such that $\mu>|z-z'|$. From (8.2) and (7.5) it follows that

$$J_2(\mu,z,z')\le C\int_{\mathcal{E}_{1,n}(\mu,z,z')}|y|^{1-n}(\mu-|y|)^{-1}|z-z'-y|^{2-n}dy = C\int_{\mathcal{E}_{1,n}(\mu,t_0(1,0\ldots0),0)}|y|^{1-n}(\mu-|y|)^{-1}|t_0(1,0,0)-y|^{2-n}dy, \tag{9.14}$$

where $t_0=|z-z'|$ and $C=\sup_{(\tilde\mu,\tilde z,\tilde z')\in(0,T)\times\partial X\times\mathbb{R}^n}\tilde\mu|\tilde z-\tilde z'|^{n-2}N(\tilde\mu,\tilde z,\tilde z')$.

Using the change of variables $y=\frac{t_0}{2}(1,0)+(s\cos(\varphi),\frac{\sqrt{s^2-t_0^2}}{2}\sin(\varphi)\omega)$ (see (9.1)), $\varphi\in(0,\pi)$, $s\in(t_0,\mu)$, $\omega\in\mathbb{S}^{n-2}$, we obtain

$$J_2(\mu,z,z')\le C'\int_{t_0}^{\mu}\int_0^{\pi}J_{2,1}(\mu,s,\varphi)d\varphi ds, \tag{9.15}$$

where $C'=2^{n-2}\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})C$ and

$$J_{2,1}(\mu,s,\varphi)=\frac{(s^2-t_0^2)^{\frac{n-3}{2}}\sin^{n-2}(\varphi)}{(s+t_0\cos(\varphi))^{n-2}(2\mu-s-t_0\cos(\varphi))(s-t_0\cos(\varphi))^{n-3}}, \tag{9.16}$$

for $\varphi\in(0,2\pi)$ and $s\in(t_0,\mu)$.

We give estimates on $J_{2,1}$. Let $\varphi\in(0,\frac{\pi}{2})$ and $s\in(t_0,\mu)$. From (9.16) and the estimates $s+t_0\cos(\varphi)\ge s$, $\sqrt{s^2-t_0^2}\sin(\varphi)\le s-t_0\cos(\varphi)$ and the estimate $2\mu-s-t_0\cos(\varphi)\ge s-t_0\cos(\varphi)$, it follows that

$$\begin{aligned} J_{2,1}(\mu,s,\varphi) &\le \frac{\sqrt{s^2-t_0^2}\sin^2(\varphi)}{s^{n-2}(s-t_0\cos(\varphi))^2}\\ &\le C_0\frac{\sqrt{s^2-t_0^2}}{s^{n-1}(s-t_0\cos(\varphi))}, \end{aligned} \tag{9.17}$$

where C_0 is defined by (5.33) (we also used the estimate $s-t_0\cos(\varphi)\ge s(1-\cos(\varphi))$). Let $\varphi\in(\frac{\pi}{2},\pi)$ and $s\in(t_0,\mu)$. From (9.16) and the estimates $s-t_0\cos(\varphi)\ge s$, $\sqrt{s^2-t_0^2}\sin(\varphi)\le s+t_0\cos(\varphi)$ and the estimate $2\mu-s-t_0\cos(\varphi)\ge s$, it follows that

$$J_{2,1}(\mu,s,\varphi)\le\frac{\sqrt{s^2-t_0^2}\sin^2(\varphi)}{s^{n-2}(s+t_0\cos(\varphi))^2}\le C_0\frac{\sqrt{s^2-t_0^2}}{s^{n-1}(s+t_0\cos(\varphi))}, \tag{9.18}$$

where C_0 is defined by (5.33) (we also used the estimate $s+t_0\cos(\varphi)\ge s(1+\cos(\varphi))$).

Combining (9.17) and (9.18) and (7.1), we obtain

$$\int_0^{\pi}J_{2,1}(\mu,s,\varphi)d\varphi\le\frac{2\pi C_0}{s^{n-1}},\ \text{for } s\in(t_0,\mu). \tag{9.19}$$

Note that

$$\begin{aligned}\int_{t_0}^{\mu}\frac{1}{s^{n-1}}ds &= \frac{1}{n-2}\left(\frac{\mu^{n-2}-t_0^{n-2}}{t_0^{n-2}\mu^{n-2}}\right)=\frac{\mu-t_0}{n-2}\sum_{i=0}^{n-3}\mu^{-1-i}t_0^{i+2-n}\\ &\le \frac{\mu-t_0}{\mu t_0^{n-2}}. \end{aligned} \tag{9.20}$$

(we used the estimate $t_0 < \mu$ which gives $\mu^{-1-i}t_0^{i+2-n} \le \mu^{-1}t_0^{n-2}$ for $i = 0 \dots n-3$).

Combining (9.15), (9.19)–(9.20), we obtain

$$J_2(\mu, z, z') \le C' \frac{\mu - t_0}{\mu t_0^{n-2}}. \tag{9.21}$$

where C' does not depend on μ and z, z'. Statement (8.5) follows from (9.21). $\square$

Proof of Lemma 8.2. Let $(\tau, x, x') \in (0,T) \times \mathbb{R}^n \times \mathbb{R}^n$ such that $\tau > |x-x'| > 0$. Set $t_0 = |x-x'|$. Let $(y_2, \dots, y_{m-1}) \in \mathcal{E}_{m-2,n}(\tau, x, x')$. Then $t_0 \le |y_2| + \dots + |y_{m-1}| + |x - x' - y_2 - \dots y_{m-1}|$. Therefore either $|x - x' - y_2 - \dots - y_{m-1}| \ge \frac{t_0}{m-1}$ or $|x - x' - y_2 - \dots - y_{m-1}| < \frac{t_0}{m-1}$ and there exists $j \in \mathbb{N}$, $j = 1 \dots n$ such that $|y_j| \ge \frac{t_0}{m-1}$. Therefore using (8.6) we obtain

$$\begin{aligned}\tilde{J}_m(\tau, x, x') &\le \sum_{j=2}^{m-1} \int_{\substack{(y_2,\dots,y_{m-1}) \in \mathcal{E}_{m-2,n}(\tau,x,x') \\ |x-x'-\sum_{i=2}^{m-1} y_i| < \frac{t_0}{m-1} \\ |y_j| \ge \frac{t_0}{m-1}}} \frac{dy_2 \dots dy_{m-1}}{|y_2|^{n-1} \dots |y_{m-1}|^{n-1} |x - x' - \sum_{i=2}^{m-1} y_i|^{n-2}} \\ &\quad + \tilde{J}_{m,0}(\tau, x, x') \\ &= (m-2)\tilde{J}_{m,1}(\tau, x, x') + \tilde{J}_{m,0}(\tau, x, x'), \end{aligned} \tag{9.22}$$

where

$$\tilde{J}_{m,0}(\tau, x, x') = \int_{\substack{(y_2,\dots,y_{m-1}) \in \mathcal{E}_{m-2,n}(\tau,x,x') \\ |x-x'-\sum_{i=2}^{m-1} y_i| > \frac{t_0}{m-1}}} \frac{dy_2 \dots dy_{m-1}}{|y_2|^{n-1} \dots |y_{m-1}|^{n-1} |x - x' - \sum_{i=2}^{m-1} y_i|^{n-2}}, \tag{9.23}$$

$$\tilde{J}_{m,1}(\tau, x, x') = \int_{\substack{(y_2,\dots,y_{m-1}) \in \mathcal{E}_{m-2,n}(\tau,x,x') \\ |x-x'-\sum_{i=2}^{m-1} y_i| < \frac{t_0}{m-1} \\ |y_2| \ge \frac{t_0}{m-1}}} \frac{dy_2 \dots dy_{m-1}}{|y_2|^{n-1} \dots |y_{m-1}|^{n-1} |x - x' - \sum_{i=2}^{m-1} y_i|^{n-2}}. \tag{9.24}$$

We first reduce the estimate of $\tilde{J}_{m,0}$ and $\tilde{J}_{m,1}$ to an estimate on

$$P_m(\tau, x, x') := \int_{(y_2,\dots,y_{m-1}) \in \mathcal{E}_{m-2,n}(\tau,x,x')} \frac{dy_2 \dots dy_{m-1}}{|y_2|^{n-1} \dots |y_{m-1}|^{n-1}}. \tag{9.25}$$

From (9.23) and the estimate $|x - x' - y_2 - \dots - y_{m-1}| > \frac{t_0}{m-1}$ it follows that

$$\tilde{J}_{m,0}(\tau, x, x') \le \left(\frac{m-1}{t_0}\right)^{n-2} P_m(\tau, x, x'). \tag{9.26}$$

From (9.24) and the estimates $|y_2| \ge \frac{t_0}{m-1} \ge |x - x' - y_2 - \dots - y_{m-1}|$ it follows that

$$\tilde{J}_{m,1}(\tau, x, x') \le \int_{\substack{(y_2,\dots,y_{m-1}) \in \mathcal{E}_{m-2,n}(\tau,x,x') \\ |x-x'-\sum_{i=2}^{m-1} y_i| < \frac{t_0}{m-1} \\ |y_2| \ge \frac{t_0}{m-1}}} \frac{dy_2 \dots dy_{m-1}}{|y_2|^{n-2} \dots |y_{m-1}|^{n-1} |x - x' - \sum_{i=2}^{m-1} y_i|^{n-1}}. \tag{9.27}$$

Therefore performing the change of variables "y_2"$= x - x' - y_2 - \dots - y_{m-1}$ we obtain

$$\tilde{J}_{m,1}(\tau, x, x') \le \tilde{J}_{m,0} \le \left(\frac{m-1}{t_0}\right)^{n-2} P_m(\tau, x, x'). \tag{9.28}$$

Now we estimate $P_3(\tau, x, x')$. From (9.1) and (9.25) it follows that

$$P_3(\tau, x, x') = \mathrm{Vol}_{n-2}(\mathbb{S}^{n-2}) \int_0^\pi \int_{t_0}^\tau \frac{\sin^{n-2}(\varphi)\,(s^2 - t_0^2)^{\frac{n-3}{2}}\,(s - t_0\cos(\varphi))}{2(s + t_0\cos(\varphi))^{n-2}} ds d\varphi. \tag{9.29}$$

Let $n=3$. Then using the estimate $\cos(\varphi)\ge -1$ (and the fact that $\frac{s-t_0\cos(\varphi)}{s+t_0\cos(\varphi)}=\frac{s-t_0}{s+t_0\cos(\varphi)}+\frac{t_0(1-\cos(\varphi))}{s+t_0\cos(\varphi))}\le 1+\frac{2t_0}{s+t_0\cos(\varphi)}$) we obtain

$$\begin{aligned}
P_3(\tau,x,x') &\le \pi\int_{t_0}^{\tau}\int_0^{\pi}\left(\sin(\varphi)-2\frac{d}{d\varphi}\ln(s+t_0\cos(\varphi))\right)d\varphi ds\\
&= 2\pi\left(\tau-t_0+\int_{t_0}^{\tau}\ln(s+t_0)ds-\int_{t_0}^{\tau}\ln(s-t_0)ds\right)\\
&= 2\pi(\tau-t_0)\left(2+\ln\left(\frac{\tau+t_0}{\tau-t_0}\right)\right) \qquad (9.30)
\end{aligned}$$

(we used the estimate $\ln(s+t_0)\le\ln(\tau+t_0)$ for $s\in(t_0,\tau)$ and we used the integral value (9.11)). Let $n\ge 4$. Using (9.29) and using the estimates $\sqrt{s^2-t_0^2}\sin(\varphi)\le s+t_0\cos(\varphi)$, $s+t_0\cos(\varphi)\ge s(1+\cos(\varphi))$ and $s-t_0\cos(\varphi)\le s+t_0\le 2s$ for $(s,\varphi)\in(t_0,\tau)\times(0,\pi)$, we obtain

$$\begin{aligned}
P_3(\tau,x,x') &\le \mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\int_0^{\pi}\int_{t_0}^{\tau}\frac{\sin^2(\varphi)\sqrt{s^2-t_0^2}(s-t_0\cos(\varphi))}{2(s+t_0\cos(\varphi))^2}dsd\varphi\\
&\le C_0\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\int_{t_0}^{\tau}\int_0^{\pi}\frac{\sqrt{s^2-t_0^2}}{s+t_0\cos(\varphi)}d\varphi ds\le \pi C_0\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})(\tau-t_0), \quad (9.31)
\end{aligned}$$

where C_0 is defined by (5.33) (we also used (7.1)).

Finally let $m\ge 4$ then

$$(y_2,\ldots,y_{m-1})\in\mathcal{E}_{m-2,n}(\tau,x,x')\Rightarrow\left(y_2\in\mathcal{E}_{1,n}(\tau,x,x')\text{ and }\sum_{i=3}^{m-1}|y_i|<\tau\right). \qquad (9.32)$$

Therefore using (9.25), spherical coordinates ("y_i"$=s_i\omega_i$, $s_i\in(0,+\infty)$, $\omega_i\in\mathbb{S}^{n-1}$ for $i=3,\ldots,m-1$) we obtain

$$\begin{aligned}
&P_m(\tau,x,x')\le P_3(\tau,x,x')\int_{|y_3|+\ldots+|y_{m-1}|<\tau}\frac{dy_3\ldots dy_{m-1}}{|y_3|^{n-1}\ldots|y_{m-1}|^{n-1}}\\
&\le P_3(\tau,x,x')\mathrm{Vol}_{n-1}(\mathbb{S}^{n-1})^{m-3}\int_{\substack{s_3+\ldots+s_{m-1}<\tau\\ 0<s_i,\ i=3,\ldots,m-1}}\Pi_{i=3}^{m-1}ds_i=P_3(\tau,x,x')\mathrm{Vol}_{n-1}(\mathbb{S}^{n-1})^{m-3}\frac{\tau^{m-3}}{(m-3)!}. \quad (9.33)
\end{aligned}$$

Finally statement (8.7) follows from (9.22), (9.26), (9.28), (9.30) and (9.33), and statement (8.8) follows from (9.22), (9.26), (9.28), (9.31) and (9.33). $\square$

Proof of Lemma 8.3. Let $n\ge 2$. Using a rotation and (8.1), we have

$$\mathrm{Vol}_n(\mathcal{E}_{1,n}(\tau,x,x'))=\mathrm{Vol}_n(\mathcal{E}_{1,n}(\tau,t_0e_1,0)), \qquad (9.34)$$

where $t_0=|x-x'|$ and $e_1=(0,\ldots,0)\in\mathbb{R}^n$.

From (9.1), it follows that

$$\mathrm{Vol}_n(\mathcal{E}_{1,n}(\tau,t_0e_1,0))=\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\int_{t_0}^{\tau}\int_0^{\pi}\left(\frac{\sin(\varphi)\sqrt{s^2-t_0^2}}{2}\right)^{n-2}\frac{s^2-t_0^2\cos^2(\varphi)}{4\sqrt{s^2-t_0^2}}dsd\varphi. \qquad (9.35)$$

From (9.35) and the estimate $\sin(\varphi)\sqrt{s^2-t_0^2} \le \sqrt{\tau^2-t_0^2}$ for $s\in(t_0,\tau)$, we obtain

$$\begin{aligned}\mathrm{Vol}_n(\mathcal{E}_{1,n}(\tau,t_0e_1,0)) &\le \mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\left(\frac{\sqrt{\tau^2-t_0^2}}{2}\right)^{n-2}\int_{t_0}^{\tau}\int_0^{\pi}\frac{s^2-t_0^2\cos^2(\phi)}{4\sqrt{s^2-t_0^2}}dsd\varphi\\ &\le \frac{1}{2}\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\left(\frac{\sqrt{\tau^2-t_0^2}}{2}\right)^{n-2}\mathrm{Vol}(\mathcal{E}_{1,2}(\tau,t_0e_1,0)). \qquad (9.36)\end{aligned}$$

We remind that $\mathrm{Vol}(\mathcal{E}_{1,2}(\tau,t_0e_1,0)) = \frac{\pi(t_0+\tau)\sqrt{\tau^2-t_0^2}}{4}$. Therefore (8.9) follows from (9.36). Lemma 8.3 is proved. □

Proof of Lemma 8.4. Let $(\mu,z,z')\in(0,T)\times\partial X\times\mathbb{R}^3$ be such that $\mu>|z-z'|>0$. Using the change of variables $y=\frac{t_0}{2}(1,0)+(s\cos(\varphi),\frac{\sqrt{s^2-t_0^2}}{2}\sin(\varphi)\omega)$ (see (9.1)), $\varphi\in(0,\pi)$, $s\in(t_0,\mu)$, $\omega\in\mathbb{S}^1$, we obtain

$$B(\mu,z,z') = \frac{\pi}{4}\int_{t_0}^{\mu}\int_0^{\pi}B_1(\mu,s,\varphi)d\varphi ds, \qquad (9.37)$$

where

$$B_1(\mu,s,\varphi) = (s^2-t_0^2\cos^2(\varphi))\sin(\varphi)\ln\left(\frac{\mu-t_0\cos(\varphi)}{\mu-s}\right), \qquad (9.38)$$

for $\varphi\in(0,2\pi)$ and $s\in(t_0,\mu)$. Using (9.38) and the estimates $\ln\left(\frac{\mu-t_0\cos(\varphi)}{\mu-s}\right)\le\ln\left(\frac{\mu+t_0}{\mu-s}\right)$, $s^2-t_0^2\cos^2(\varphi)\le\mu^2$, we obtain

$$\int_0^{\pi}B_1(\mu,s,\varphi)d\varphi \le \mu^2\int_0^{\pi}\sin(\varphi)d\varphi\ln\left(\frac{\mu+t_0}{\mu-s}\right), \qquad (9.39)$$

for $s\in(t_0,\mu)$. Combining (9.37), (9.39) and (9.11) we obtain

$$B(\mu,z,z') \le \frac{\mu^2\pi}{2}(\mu-t_0)\left(\ln\left(\frac{\mu+t_0}{\mu-t_0}\right)+1\right), \qquad (9.40)$$

which proves (8.11). □

10 The distributional kernel of the operators H_m and the proof of Proposition 3.1

Before we prove Proposition 3.1 we shall introduce and prove Proposition 10.1 given below, which gives the distributional kernel of the operators H_m defined by (2.12).

Let $\bar{E}$ denotes the nonnegative mesurable function from $\mathbb{R}^n\times\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$\bar{E}(x_1,x_2) = e^{-\int_0^{|x_1-x_2|}\sigma(x_1-s\frac{x_1-x_2}{|x_1-x_2|},\frac{x_1-x_2}{|x_1-x_2|})ds}\Theta(x_1,x_2),\ \text{for a.e. } (x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n, \qquad (10.1)$$

where Θ is defined by (2.10). For $m\ge3$, we define recursively the nonnegative measurable real function $\bar{E}(x_1,\ldots,x_m)$ by the formula

$$\bar{E}(x_1,\ldots,x_m) = \bar{E}(x_1,\ldots,x_{m-1})\bar{E}(x_{m-1},x_m), \qquad (10.2)$$

for $(x_1,\ldots,x_m)\in(\mathbb{R}^n)^m$.

Concerning the distributional kernel of the H_m, $m\ge2$, we have the following result.

Proposition 10.1. *We have*

$$H_m(t)\phi(x,v)=\int_{X\times\mathbb{S}^{n-1}}\beta_m(t,x,v,x',v')\phi(x',v')dx'dv', \tag{10.3}$$

for $t\in(0,T)$ and a.e. $(x,v)\in X\times\mathbb{S}^{n-1}$ and for $m\geq 2$, where

$$\begin{aligned}\beta_2(t,x,v,x',v') &= \int_0^t \chi_{(0,t-s_2)}(|x'-(x-s_2v')|)\frac{2^{n-2}\left(t-s_2-(x-s_2v'-x')\cdot v\right)^{n-3}}{|x-s_2v'-x'-(t-s_2)v|^{2n-4}}\\ &\quad\times\left[\bar{E}(x,x-(t-s_1-s_2)v,x'+s_2v',x')k(x-(t-s_1-s_2)v,v_1,v)\right.\\ &\quad\left.\times k(x'+s_2v',v',v_1)\right]_{\substack{v_1=\frac{x-s_2v'-x'-(t-s_1-s_2)v}{s_1}\\ s_1=\frac{|x-s_2v'-x'-(t-s_2)v|^2}{2(t-s_2-(x-x'-s_2v')\cdot v)}}} ds_2,\end{aligned} \tag{10.4}$$

for $t\in(0,T)$ and a.e. $(x,v,x',v')\in X\times\mathbb{S}^{n-1}\times X\times\mathbb{S}^{n-1}$, and where

$$\begin{aligned}&\beta_m(t,x,v,x',v')=\int_{(\mathbb{S}^{n-1})^{m-2}}\int_{\substack{s_2+\ldots+s_m\leq t\\ s_i\geq 0,\ i=2\ldots m}}\chi_{(0,t-s_m-\ldots-s_2)}(|x'+s_mv'+\ldots+s_2v_2-x|)\\ &\times\frac{2^{n-2}\left(t-s_2-\ldots-s_m-(x-x'-s_2v_2-\ldots-s_{m-1}v_{m-1}-s_mv')\cdot v\right)^{n-3}}{|x-x'-s_2v_2-\ldots-s_{m-1}v_{m-1}-s_mv'-(t-s_2-\ldots-s_m)v|^{2n-4}}\left[\bar{E}(x,x-(t-s_1-\ldots-s_m)v,\right.\\ &x'+s_mv'+s_{m-1}v_{m-1}+\ldots+s_2v_2,x'+s_mv'+s_{m-1}v_{m-1}+\ldots+s_3v_3,\\ &\ldots,x'+s_mv',x')k(x-(t-s_1-\ldots-s_m)v,v_1,v)k(x'+s_mv'+s_{m-1}v_{m-1}+\ldots+s_2v_2,v_2,v_1)\\ &\ldots k(x'+s_mv'+s_{m-1}v_{m-1}+\ldots+s_{i+1}v_{i+1},v_{i+1},v_i)\ldots\\ &\left.k(x'+s_mv',v',v_{m-1})\right]_{\substack{v_1=\frac{x-x'-s_2v_2-\ldots-s_{m-1}v_{m-1}-s_mv'-(t-s_1-\ldots-s_m)v}{s_1}\\ s_1=\frac{|x-x'-s_2v_2-\ldots-s_{m-1}v_{m-1}-s_mv'-(t-s_2-\ldots-s_m)v|^2}{2(t-s_2-\ldots-s_m-(x-x'-s_2v_2-\ldots s_{m-1}v_{m-1}-s_mv')\cdot v)}}}\ ds_2\ldots ds_mdv_2\ldots dv_{m-1},\end{aligned} \tag{10.5}$$

for $t\in(0,T)$ and a.e. $(x,v,x',v')\in X\times\mathbb{S}^{n-1}\times X\times\mathbb{S}^{n-1}$, $m\geq 3$.

Proof of Proposition 10.1. Note that

$$\begin{aligned}H_2(t)\phi(x,v) &= \left(\int_0^t\int_0^{t-s_1}U_1(t-s_1-s_2)A_2U_1(s_1)A_2U_1(s_2)\phi ds_2ds_1\right)(x,v)\\ &= \left(\int_0^t\left(\int_0^{t-s_2}U_1(t-s_1-s_2)A_2U_1(s_1)A_2ds_1\right)U_1(s_2)\phi ds_2\right)(x,v)\\ &= \int_0^t\int_0^{t-s_2}\bar{E}(x,x-(t-s_1-s_2)v)\int_{\mathbb{S}^{n-1}}k(x-(t-s_1-s_2)v,v_1,v)\\ &\quad\times\bar{E}(x-(t-s_1-s_2)v,x-(t-s_1-s_2)v-s_1v_1)\\ &\quad\times\int_{\mathbb{S}^{n-1}}k(x-(t-s_2-s_1)v-s_1v_1,v_2,v_1)\\ &\quad\times\bar{E}(x-(t-s_1-s_2)v-s_1v_1,x-(t-s_1-s_2)v-s_1v_1-s_2v_2)\\ &\quad\times\phi(x-(t-s_1-s_2)v-s_1v_1-s_2v_2,v_2)dv_2dv_1ds_1ds_2,\end{aligned}$$

for $t\in(0,T)$ and $(x,v)\in X\times\mathbb{S}^{n-1}$, where functions $\bar{E}$ are defined by (10.1)–(10.2).

Using the change of variables "$y(s_1,v_1)=(t-s_2-s_1)v+s_1v_1$" we obtain

$$\begin{aligned}H_2(t)\phi(x,v) &= \int_0^t\int_{\mathbb{S}^{n-1}}\left[\bar{E}(x,x-(t-s_1-s_2)v,x-y,x-y-s_2v_2)k(x-(t-s_1-s_2)v,v_1,v)\right.\\ &\quad\left.\times k(x-y,v_2,v_1)\right]_{\substack{v_1=\frac{y-(t-s_1-s_2)v}{s_1}\\ s_1=\frac{|y-(t-s_2)v|^2}{2(t-s_2-y\cdot v)}}}\\ &\quad\times\frac{2^{n-2}\left((t-s_2)-y\cdot v\right)^{n-3}}{|y-(t-s_2)v|^{2n-4}}\phi(x-y-s_2v_2,v_2)dydv_2ds_2.\end{aligned}$$

Hence we obtain (10.3). Note that

$$
\begin{aligned}
(H_3(t)\phi)(x,v) &= \int_0^t H_2(t-s_3)A_2U_1(s_3)\phi ds_3 \\
&= \int_0^t \int_{X\times\mathbb{S}^{n-1}} \beta_2(t-s_3,x,v,x_2,v_2)(A_2U_1(s_3))\phi(x_2,v_2)dx_2dv_2ds_3 \\
&= \int_{X\times\mathbb{S}^{n-1}} \int_0^t \beta_2(t-s_3,x,v,x_2,v_2)\int_{\mathbb{S}^{n-1}} k(x_2,v',v_2)\bar{E}(x_2,x_2-s_3v') \\
&\quad \times\phi(x_2-s_3v',v')dv'ds_3dx_2dv_2.
\end{aligned}
$$

Hence

$$
(H_3(t)\phi)(x,v) = \int_{X\times\mathbb{S}^{n-1}} \beta_3(t,x,v,x',v')\phi(x',v')dx'dv', \tag{10.6}
$$

where

$$
\begin{aligned}
\beta_3(t,x,v,x',v') &= \int_{\mathbb{S}^{n-1}}\int_0^t\int_0^{t-s_3} \chi_{(0,t-s_3-s_2)}(|x'+s_3v'-x+s_2v_2|) \\
&\quad\times\frac{2^{n-2}\left(t-s_2-s_3-(x-s_2v_2-x'-s_3v')\cdot v\right)^{n-3}}{|x-x'-s_2v_2-s_3v'-(t-s_2-s_3)v|^{2n-4}} \\
&\quad\times\big[\bar{E}(x,x-(t-s_1-s_2-s_3)v,x'+s_2v_2+s_3v',x'+s_3v',x') \\
&\quad\times k(x-(t-s_3-s_2-s_1)v,v_1,v)k(x'+s_2v_2+s_3v',v_2,v_1) \\
&\quad\times\, k(x'+s_3v',v',v_2)\big]_{\substack{v_1=\frac{x-x'-s_2v_2-s_3v'-(t-s_1-s_2-s_3)v}{s_1}\\ s_1=\frac{|x-x'-s_2v_2-s_3v'-(t-s_2-s_3)v|^2}{2(t-s_2-s_3-(x-x'-s_2v_2-s_3v')\cdot v)}}} ds_2ds_3dv_2.
\end{aligned}
\tag{10.7}
$$

The proof of (10.5) follows by induction from (10.6) and (2.13). □

Proof of (8.12)–(8.13). We recall that

$$
\left(A_2G_-(s)\phi_S\right)(z,w) = \int_{\partial X}\left[k(z,v',w)S(x',v')|\nu(x')\cdot v'|\right]_{v'=\frac{z-x'}{|z-x'|}}\frac{E(z,x')}{|z-x'|^{n-1}}\phi(s-|z-x'|,x')d\mu(x'), \tag{10.8}
$$

for a.e. $(z,w)\in X\times\mathbb{S}^{n-1}$ and $\phi\in L^1((0,\eta)\times\partial X)$ (see the derivation of (3.9) and (3.10) given in Section 3).

Let $m=2$. Then from (2.12) and (2.13) it follows that

$$
\begin{aligned}
&A_{2,S,W}(\phi)(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}}(\nu(x)\cdot v)W(x,v)\int_{-\infty}^t\int_0^{t-s}\int_{\mathbb{S}^{n-1}}\int_{\partial X}[k(x-(t-s-s_1)v,v_1,v) \\
&\times k(x-(t-s-s_1)v-s_1v_1,v',v_1)S(x',v')|\nu(x')\cdot v'|]_{v'=\frac{x-(t-s-s_1)v-s_1v_1-x'}{|x-(t-s-s_1)v-s_1v_1-x'|}} E(x,x-(t-s-s_1)v, \\
&x-(t-s-s_1)v-s_1v_1,x')\frac{\phi(s-|x-(t-s-s_1)v-s_1v_1-x'|,x')}{|x-(t-s-s_1)v-s_1v_1-x'|^{n-1}}d\mu(x')dv_1ds_1dsdv.
\end{aligned}
$$

Performing the change of variables $y(s_1,v_1)=(t-s-s_1)v+s_1v_1$, we obtain

$$
\begin{aligned}
&A_{2,S,W}(\phi)(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}\times\partial X\times\mathbb{R}^n}(\nu(x)\cdot v)W(x,v)\int_{-\infty}^t\chi_{(0,t-s)}(|y|) \\
&\times[E(x,x-(t-s-s_1)v,x-y,x')k(x-(t-s-s_1)v,v_1,v)k(x-y,v',v_1)S(x',v') \\
&\times|\nu(x')\cdot v'|]_{\substack{s_1=\frac{|(t-s)v-y|^2}{2(t-s-y\cdot v)}\\ v_1=\frac{y-(t-s-s_1)v}{s_1}\\ v'=\frac{x-y-x'}{|x-y-x'|}}}\frac{2^{n-2}(t-s-y\cdot v)^{n-3}\phi(s-|x-y-x'|,x')}{|(t-s)v-y|^{2n-4}|x-y-x'|^{n-1}}dsdyd\mu(x')dv.
\end{aligned}
\tag{10.9}
$$

Performing the change of variables "y"$= x - x' - y$ and $t' = s - |y|$ we obtain (8.12).

Let $m = 3$. Then from (3.5), (10.3) and (10.8) it follows that

$$A_{3,S,W}(\phi)(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x)\cdot v) W(x,v) \int_{-\infty}^{t} \int_{X\times\mathbb{S}^{n-1}} \beta_2(t-s,x,v,x_2,v_2) \tag{10.10}$$

$$\int_{\partial X} [k(x_2,v',v_2)S(x',v')|\nu(x')\cdot v'|]_{v'=\frac{x_2-x'}{|x_2-x'|}} \frac{E(x_2,x')}{|x_2-x'|^{n-1}} \phi(s-|x_2-x'|,x')d\mu(x')dx_2dv_2dsdv,$$

for $t\in(0,T)$ and $x\in\partial X$. From (10.10) and (10.4) we obtain

$$A_{3,S,W}(\phi)(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x)\cdot v) W(x,v) \int_{X\times\mathbb{S}^{n-1}\times\partial X} \int_{-\infty}^{t} \int_{0}^{t-s} \chi_{(0,t-s-s_2)}(|x_2-(x-s_2v_2)|)$$

$$\frac{2^{n-2}\left(t-s-s_2-(x-s_2v_2-x_2)\cdot v\right)^{n-3}}{|x_2-x'|^{n-1}\left|x-s_2v_2-x_2-(t-s-s_2)v\right|^{2n-4}}$$

$$\times [E(x,x-(t-s_1-s_2)v,x_2+s_2v_2,x_2,x')k(x-(t-s_1-s_2)v,v_1,v)k(x_2+s_2v_2,v_2,v_1)$$

$$k(x_2,v',v_2)S(x',v')|\nu(x')\cdot v'|]_{\substack{v_1=\frac{x-s_2v_2-x_2-(t-s-s_1-s_2)v}{s_1}\\ s_1=\frac{|x-s_2v_2-x_2-(t-s-s_2)v|^2}{2(t-s-s_2-(x-x_2-s_2v_2)\cdot v)}\\ v'=\frac{x_2-x'}{|x_2-x'|}}} \phi(s-|x_2-x'|,x')ds_2dsdx_2dv_2d\mu(x')dv.$$

Performing the change of variables $y_2 = s_2v_2$ and $y_3 = x_2 - x'$ we obtain (8.13) for "$m = 3$".

Let $m \geq 3$. From (3.5), (10.3), (10.5) and (10.8) it follows that

$$\begin{aligned} A_{m+1,S,W}(\phi)(t,x) &= \int_{\mathbb{S}^{n-1}_{x,+}} \int_{\partial X} (\nu(x)\cdot v)W(x,v) \int_X \int_{(-\infty,t-|x_m-x'|)\times\mathbb{S}^{n-1}} \beta_m(t-t'-|x_m-x'|,x,v,x_m,v_m) \\ &\quad [k(x_m,v',v_m)S(x',v')|\nu(x')\cdot v'|]_{v'=\frac{x_m-x'}{|x_m-x'|}} \frac{E(x_m,x')}{|x_m-x'|^{n-1}}\phi(t',x')d\mu(x')dt'dx_mdv_mdv \\ &= \int_{(0,\eta)\times\partial X} \gamma_{m+1}(t-t',x,x')\phi(t',x')dt'd\mu(x'), \end{aligned} \tag{10.11}$$

where

$$\gamma_{m+1}(\tau,x,x') := \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x)\cdot v)W(x,v) \int_{X\times\mathbb{S}^{n-1}} \chi_{(0,+\infty)}(\tau-|x_m-x'|)$$

$$\int_{(\mathbb{S}^{n-1})^{m-2}} \int_{\substack{s_2+\ldots+s_m\leq\tau-|x_m-x'|\\ s_i\geq 0,\ i=2\ldots m}} \chi_{(0,\tau-|x_m-x'|-s_m-\ldots-s_2)}(|x_m+s_mv_m+\ldots+s_2v_2-x|)$$

$$\times\frac{2^{n-2}\left(\tau-|x_m-x'|-s_2-\ldots-s_m-(x-x_m-s_2v_2-\ldots-s_mv_m)\cdot v\right)^{n-3}}{|x_m-x'|^{n-1}|x-x_m-s_2v_2-\ldots-s_mv_m-(\tau-|x_m-x'|-s_2-\ldots-s_m)v|^{2n-4}}$$

$$\times [E(x,x-(\tau-|x_m-x'|-s_1-\ldots-s_m)v,x_m+s_mv_m\ldots+s_2v_2,x_m+s_mv_m+\ldots+s_3v_3,\ldots,$$

$$x_m+s_mv_m,x_m,x')k(x-(\tau-|x_m-x'|-s_1-\ldots-s_m)v,v_1,v)k(x_m+s_mv_m+\ldots+s_2v_2,v_2,v_1)$$

$$\ldots k(x_m+s_mv_m+\ldots+s_{i+1}v_{i+1},v_{i+1},v_i)\ldots$$

$$k(x_m+s_mv_m,v_m,v_{m-1})k(x_m,v',v_m)S(x',v')|\nu(x')\cdot v'|]_{\substack{v_1=\frac{x-x_m-s_2v_2-\ldots-s_mv_m-(\tau-|x_m-x'|-s_1-\ldots-s_m)v}{s_1}\\ s_1=\frac{|x-x_m-s_2v_2-\ldots-s_mv_m-(\tau-|x_m-x'|-s_2-\ldots-s_m)v|^2}{2(t-s_2-\ldots-s_m-(x-x'-s_2v_2-\ldots s_mv_m)\cdot v)}\\ v'=\frac{x_m-x'}{|x_m-x'|}}}$$

$$ds_2\ldots ds_mdv_2\ldots dv_{m-1}dx_mdv_mdv. \tag{10.12}$$

Performing the change of variables $y_i = s_iv_i$, $i = 2\ldots m$, and $y_{m+1} = x_{m+1} - x'$, we obtain (8.13) for "$m \geq 4$". $\square$

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